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Finite element approximation of eigenvalue problems

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We discuss the finite element approximation of eigenvalue problems associated with compact operators. While the main emphasis is on symmetric problems, some comments are present for non-self-adjoint operators as well. The topics covered include standard Galerkin approximations, non-conforming approximations, and approximation of eigenvalue problems in mixed form. Some applications of the theory are presented and, in particular, the approximation of the Maxwell eigenvalue problem is discussed in detail. The final part tries to introduce the reader to the fascinating setting of differential forms and homological techniques with the description of the Hodge–Laplace eigenvalue problem and its mixed equivalent formulations. Several examples and numerical computations complete the paper, ranging from very basic exercises to more significant applications of the developed theory.

CONTENTS

1. Introduction

The aim of this paper is to provide the reader with an overview of the state of the art in the numerical analysis of the finite element approximation of eigenvalue problems arising from partial differential equations.

The work consists of four parts, which are ordered according to their increasing difficulty. The material is arranged in such a way that it should be possible to use it (or part of it) as a reference for a graduate course.

Part 1 presents several examples and reports on some academic numerical computations. The results presented range from a very basic level (such as the approximation of the one-dimensional Laplace operator), suited to those just starting work in this subject, to more involved examples. In particular, we give a comprehensive review of the Galerkin approximation of the Laplace eigenvalue problem (also in the presence of a singular domain and of non-conforming schemes), of the mixed approximation of the Laplace eigenvalue problem (with stable or unstable schemes), and of the Maxwell eigenvalue problem. Some of the presented material is new, in particular, the numerical results for the one-dimensional mixed Laplacian with the $\mathcal{P}_1 - \mathcal{P}_1$ and the $\mathcal{P}_2 - \mathcal{P}_0$ scheme.

Part 2 contains the main core of the theory concerning the Galerkin approximation of variationally posed eigenvalue problems. With a didactic purpose, we included a direct proof of convergence for the eigenvalues and eigenfunctions of the Laplace equation approximated with piecewise linear elements. By direct proof, we mean a proof which does not make use of the abstract spectral approximation theory, but is based on basic properties of the Rayleigh quotient. This proof is not new, but particular care has been paid to the analysis of the case of multiple eigenfunctions. In Section 9 we describe the so-called Babuška–Osborn theory. As an example of application we analyse the approximation of the eigensolutions of an elliptic operator. Then, we provide another application which involves the non-conforming Crouzeix–Raviart element for the approximation of the Laplace eigenvalue problem. The results of this section are probably not new, but we could not find a reference providing a complete analysis of this form.

Part 3 is devoted to the approximation theory of eigenvalue problems in mixed form. We recall that the natural conditions for the well-posedness and stability of source mixed problems (the classical inf-sup conditions) are not good hypotheses for convergence of the eigensolutions. It is standard to consider two different mixed formulations: problems of the first type (also known as $(f, 0)$ problems) and of the second type $(0, g)$. The first family is used, for instance, when the Stokes system is considered, and an example of an application for the second one is the mixed Laplace eigenvalue problem. The sufficient and necessary conditions for the convergence of eigenvalues and eigenfunctions of either type of mixed problem are discussed.

Finally, Part 4 deals with the homological techniques which lead to the finite element exterior calculus. We recall the Hodge–Laplace eigenvalue problem and show the links between this problem in the language of differential forms and standard eigenvalue problems for differential operators. In particular, we study the Maxwell eigenvalue problem and discuss the main tools for its analysis.

In a project like this one, it is responsibility of the author to make some choices about the material to be included. We acknowledge that we would have added some more subjects, but finally we had to trim our original plan. In particular, we completely ignored the topic of *a posteriori* and adaptive error analysis for eigenvalue problems. For this active and fundamental research field the reader is referred to the following papers and to the references therein: Hackbusch (1979), Larson (2000), Morin, Nochetto and Siebert (2000), Heuveline and Rannacher (2001), Neymeyr (2002), Durán, Padra and Rodríguez (2003), Gardini (2004), Carstensen (2008), Giani and Graham (2009), Grubišić and Ovall (2009) and Garau, Morin and Zuppa (2009). The p and hp version of finite elements is pretty much related to this topic: we give some references on this issue in Section 20 for the approximation of Maxwell's eigenvalue problem. Another area that deserves attention is the discontinuous Galerkin approximation of eigenvalue problems. We refer to the following papers and to the references therein: Hesthaven and Warburton (2004), Antonietti, Buffa and Perugia (2006), Buffa and Perugia (2006), Warburton and Embree (2006), Creusé and Nicaise (2006), Buffa, Houston and Perugia (2007) and Brenner, Li and Sung (2008). Nonstandard approximations, including mimetic schemes (Cangiani, Gardini and Manzini 2010), have not been discussed. Another important result we did not include deals with the approximation of non-compact operators (Descloux, Nassif and Rappaz 1978*a*, 1978*b*). It is interesting to note that such results have often been used for the analysis of the non-conforming approximation of compact operators and, in particular, of the approximation of Maxwell's eigenvalue problem.

Throughout this paper we quote in each section the references we need. We tried to include all significant references we were aware of, but obviously many others have not been included. We apologize for that in advance and encourage all readers to inform the author about results that would have deserved more discussion.

PART ONE
Some preliminary examples

In this section we discuss some numerical results concerning the finite element approximation of eigenvalue problems arising from partial differential equations. The presented examples provide motivation for the rest of this survey and will be used for the applications of the developed theory. We only consider *symmetric* eigenvalue problems, so we are looking for *real* eigenvalues.

2. The one-dimensional Laplace eigenvalue problem

We start with a very basic and well-known one-dimensional example. Let Ω be the open interval $]0,\pi[$ and consider the problem of finding eigenvalues λ and eigenfunctions u with $u \neq 0$ such that

$$-u''(x) = \lambda u(x) \qquad \text{in } \Omega, \tag{2.1a}$$

$$u(0) = u(\pi) = 0. \tag{2.1b}$$

It is well known that the eigenvalues are given by the squares of the integer numbers $\lambda = 1, 4, 9, 16, \ldots$ and that the corresponding eigenspaces are spanned by the eigenfunctions $\sin(kx)$ for $k = 1, 2, 3, 4 \ldots$. A standard finite element approximation of problem (2.1) is obtained by considering a suitable variational formulation. Given $V = H_0^1(\Omega)$, multiplying our equation by $v \in V$, and integrating by parts, yields the following: find $\lambda \in \mathbb{R}$ and a non-vanishing $u \in V$ such that

$$\int_0^\pi u'(x)v'(x)\,\mathrm{d}x = \lambda \int_0^\pi u(x)v(x)\,\mathrm{d}x \quad \forall v \in V. \tag{2.2}$$

A Galerkin approximation of this variational formulation is based on a finite-dimensional subspace $V_h = \mathrm{span}\{\varphi_1, \ldots, \varphi_N\} \subset V$, and consists in looking for discrete eigenvalues $\lambda_h \in \mathbb{R}$ and non-vanishing eigenfunctions $u_h \in V_h$ such that

$$\int_0^\pi u_h'(x)v'(x)\,\mathrm{d}x = \lambda_h \int_0^\pi u_h(x)v(x)\,\mathrm{d}x \quad \forall v \in V_h.$$

It is well known that this gives an algebraic problem of the form

$$A\mathsf{x} = \lambda M\mathsf{x},$$

where the stiffness matrix $A = \{a_{ij}\}_{i,j=1}^N$ is defined as

$$a_{ij} = \int_0^\pi \varphi_j'(x)\varphi_i'(x)\,\mathrm{d}x$$

and the mass matrix $M = \{m_{ij}\}_{i,j=1}^N$ is

$$m_{ij} = \int_0^\pi \varphi_j(x)\varphi_i(x)\,\mathrm{d}x.$$

Given a uniform partition of $[0,\pi]$ of size h, let V_h be the space of continuous piecewise linear polynomials vanishing at the end-points (standard conforming $\mathcal{P}_1$ finite elements); then the associated matrices read

$$a_{ij} = \frac{1}{h}\cdot\begin{cases} 2 & \text{for } i=j,\\ -1 & \text{for } |i-j|=1,\\ 0 & \text{otherwise,}\end{cases} \qquad m_{ij} = h\cdot\begin{cases} 4/6 & \text{for } i=j,\\ 1/6 & \text{for } |i-j|=1,\\ 0 & \text{otherwise,}\end{cases}$$

with $i,j = 1,\dots,N$, where the dimension N is the number of internal nodes in the interval $[0,\pi]$. It is well known that in this case it is possible to compute the explicit eigenmodes: given $k \in \mathbb{N}$, the kth eigenspace is generated by the interpolant of the continuous solution

$$u_h^{(k)}(ih) = \sin(kih), \quad i = 1,\dots,N, \tag{2.3}$$

and the corresponding eigenvalue is

$$\lambda_h^{(k)} = (6/h^2)\frac{1-\cos kh}{2+\cos kh}. \tag{2.4}$$

It is then immediate to deduce the optimal estimates (as $h \to 0$)

$$\|u^{(k)} - u_h^{(k)}\|_V = O(h) \qquad |\lambda^{(k)} - \lambda_h^{(k)}| = O(h^2) \tag{2.5}$$

with $u^{(k)}(x) = \sin(kx)$ and $\lambda^{(k)} = k^2$.

We would like to make some comments about this first example. Although here the picture is very simple and widely known, some of the observations generalize to more complicated situations and will follow from the abstract theory, which is the main object of this survey.

First of all, it is worth noticing that, even if not explicitly stated, estimates (2.5) depend on k. In particular, the estimate on the eigenvalues can be made more precise by observing that

$$\lambda_h^{(k)} = k^2 + (k^4/12)h^2 + O(k^6h^4), \quad \text{as } h \to 0. \tag{2.6}$$

This property has a clear physical meaning: since the eigenfunctions present more and more oscillations when the frequency increases, an increasingly fine mesh is required to keep the approximation error within the same accuracy.

The second important consequence of (2.4) is that all eigenvalues are approximated from above. This behaviour, which is related to the so-called min-max property (see Proposition 7.2), can be stated as follows:

$$\lambda^{(k)} \le \lambda_h^{(k)} \le \lambda^{(k)} + C(k)h^2.$$

The first estimate in (2.5) on the convergence of the eigenfunctions requires some additional comments. It is clear that the solution of the algebraic system arising from (2.2) does not give, in general, the eigenfunctions described in (2.3). Since in this simple example all eigenspaces are one-dimensional, we might expect that the numerical solver will provide us with multiples of the functions in (2.3). It is evident that if we want to perform an automatic error estimation, a first step will be to normalize the computed eigenfunctions so that they have the same norm as the continuous ones. This, however, is not enough, since there can be a difference in sign, so we have to multiply them by ± 1 in order for the scalar product with the continuous eigenfunctions to be positive.

Remark 2.1. If the same eigenvalue computation is performed with V_h equal to the space of continuous piecewise polynomials of degree at most p and vanishing at the end-points (standard conforming $\mathcal{P}_p$ finite elements), then estimates (2.5) become

$$\|u^{(k)} - u_h^{(k)}\|_V = O(h^p) \qquad |\lambda^{(k)} - \lambda_h^{(k)}| = O(h^{2p}).$$

In any case, the order of approximation for the eigenvalues is double with respect to the approximation rate of the corresponding eigenfunctions. This is the typical behaviour of *symmetric* eigenvalue problems.

3. Some numerical results for the two-dimensional Laplace eigenvalue problem

In this section we present some numerical results for the Laplace eigenvalue problem in two dimensions. We use different domains and finite elements.

Given an open Lipschitz domain $\Omega \subset \mathbb{R}^2$, we are interested in the following problem: find eigenvalues λ and eigenfunctions u with $u \neq 0$ such that

$$-\Delta u(x,y) = \lambda u(x,y) \quad \text{in } \Omega, \tag{3.1a}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{3.1b}$$

Given $V = H_0^1(\Omega)$, a variational formulation of (3.1) can be obtained as follows: find $\lambda \in \mathbb{R}$ and $u \in V$, with $u \neq 0$, such that

$$\int_\Omega \mathbf{grad}\, u(x,y) \cdot \mathbf{grad}\, v(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lambda \int_\Omega u(x,y)v(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \forall v \in V.$$

A Galerkin approximation based on a finite-dimensional subspace $V_h \subset V$ then reads: find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$, with $u_h \neq 0$, such that

$$\int_\Omega \mathbf{grad}\, u_h(x,y) \cdot \mathbf{grad}\, v(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lambda_h \int_\Omega u_h(x,y)v(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \forall v \in V_h.$$

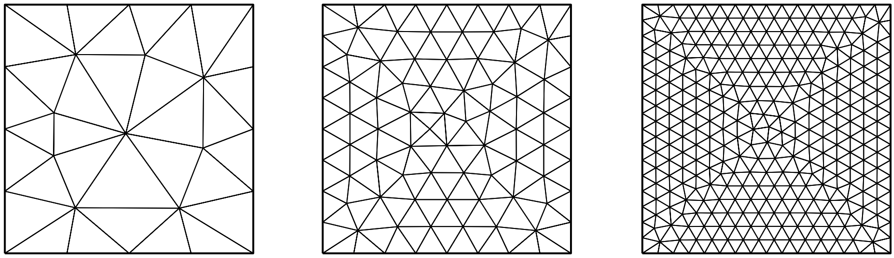

Figure 3.1. Sequence of unstructured meshes ($N = 4, 8, 16$).

3.1. The Laplace eigenvalue problem on the square: continuous piecewise linears

Let Ω be the square $]0, \pi[\times]0, \pi[$. It is well known that the eigenvalues of (3.1) are given by $\lambda_{m,n} = m^2 + n^2$ (with m and n strictly positive integers) and the corresponding eigenfunctions are $u_{m,n} = \sin(mx)\sin(ny)$. Throughout this subsection we are going to use continuous piecewise linear finite elements on triangles.

Our first computation involves a standard sequence of regular unstructured meshes, which is shown in Figure 3.1. Table 3.1 lists the first ten computed eigenvalues and their rate of convergence towards the exact values. It is evident that the scheme is convergent and that the convergence is quadratic. The abstract theory we are going to present will show that the eigenfunctions are first-order convergent in V.

Moreover, from Table 3.1 we can see behaviour similar to that observed in the one-dimensional example: all eigenvalues are approximated from above and the relative error increases with the rank of the eigenvalues in the spectrum (for instance, on the finest mesh, the relative error for the 10th eigenvalue is more than eight times the error for the first one).

This two-dimensional example allows us to make some important comments on multiple eigenvalues. If we look, for instance, at the double eigenvalue $\lambda = 5$, we see that there are two *distinct* discrete eigenvalues $\lambda_h^{(2)} < \lambda_h^{(3)}$ approximating it. Both eigenvalues are good approximations of the exact solution, and on the finest mesh their difference is smaller than 10^{-4}. A natural question concerns the behaviour of the corresponding eigenfunctions. The answer to this question is not trivial: indeed, the exact eigenspace has dimension equal to 2 and it is spanned by the functions $u_{1,2} = \sin x \sin(2y)$ and $u_{2,1} = \sin(2x)\sin y$. On the other hand, since the discrete eigenvalues are distinct, the approximating eigenspace consists of two separate one-dimensional eigenspaces. In particular, we cannot expect an estimate similar to the first one of (2.5) (even after normalization and choice of the sign for each discrete eigenfunction), since there is no rea-

Table 3.1. Eigenvalues computed on the unstructured mesh sequence.

Exact	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.2468	2.0463 (2.4)	2.0106 (2.1)	2.0025 (2.1)	2.0006 (2.0)
5	6.5866	5.2732 (2.5)	5.0638 (2.1)	5.0154 (2.0)	5.0038 (2.0)
5	6.6230	5.2859 (2.5)	5.0643 (2.2)	5.0156 (2.0)	5.0038 (2.0)
8	10.2738	8.7064 (1.7)	8.1686 (2.1)	8.0402 (2.1)	8.0099 (2.0)
10	12.7165	11.0903 (1.3)	10.2550 (2.1)	10.0610 (2.1)	10.0152 (2.0)
10	14.3630	11.1308 (1.9)	10.2595 (2.1)	10.0622 (2.1)	10.0153 (2.0)
13	19.7789	14.8941 (1.8)	13.4370 (2.1)	13.1046 (2.1)	13.0258 (2.0)
13	24.2262	14.9689 (2.5)	13.4435 (2.2)	13.1053 (2.1)	13.0258 (2.0)
17	34.0569	20.1284 (2.4)	17.7468 (2.1)	17.1771 (2.1)	17.0440 (2.0)
17		20.2113	17.7528 (2.1)	17.1798 (2.1)	17.0443 (2.0)
DOF	9	56	257	1106	4573

son why, for instance, the eigenspace associated to $\lambda_h^{(2)}$ should provide a good approximation of $u_{1,2}$. The right approach to this problem makes use of the direct sum of the eigenspaces corresponding to $\lambda_h^{(2)}$ and $\lambda_h^{(3)}$, that is, $\mathrm{span}\{u_h^{(2)}, u_h^{(3)}\}$, which does in fact provide a good approximation to the two-dimensional eigenspace associated with $\lambda = 5$. The definition of such an approximation, which relies on the notion of a *gap* between Hilbert spaces, will be made more precise later on. For the moment, we make explicit the concept of convergence in this particular situation which can be stated as follows: there exist constants $\alpha_{1,2}(h)$, $\alpha_{2,1}(h)$, $\beta_{1,2}(h)$ and $\beta_{2,1}(h)$ such that

$$\begin{aligned} &\|u_{1,2} - \alpha_{1,2}(h)u_h^{(2)} - \beta_{1,2}(h)u_h^{(3)}\|_V = O(h),\\ &\|u_{2,1} - \alpha_{2,1}(h)u_h^{(2)} - \beta_{2,1}(h)u_h^{(3)}\|_V = O(h). \end{aligned} \tag{3.2}$$

It should be clear that the way $u_{1,2}$ and $u_{2,1}$ influence the behaviour of $u_h^{(2)}$ and $u_h^{(3)}$ is mesh-dependent: on the unstructured mesh sequences used for our computations, we cannot expect the α's and the β's to stabilize towards fixed numbers. In order to demonstrate this phenomenon, we display in Figure 3.2 the computed eigenfunctions associated with $\lambda_h^{(2)}$ for $N = 8$, 16, and 32. The corresponding plot for the computed eigenfunctions associated with $\lambda_h^{(3)}$ is shown in Figure 3.3. For the sake of comparison, the exact eigenfunctions $u_{1,2}$ and $u_{2,1}$ are plotted in Figure 3.4.

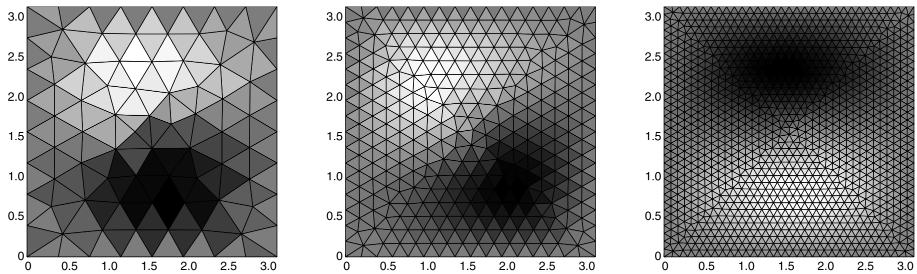

Figure 3.2. Eigenfunction associated with $\lambda_h^{(2)}$ on the unstructured mesh sequence.

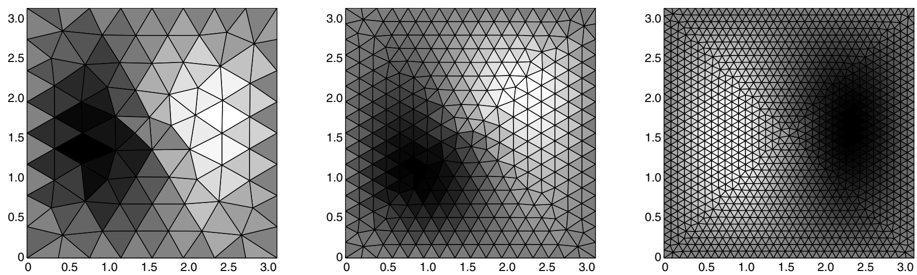

Figure 3.3. Eigenfunction associated with $\lambda_h^{(3)}$ on the unstructured mesh sequence.

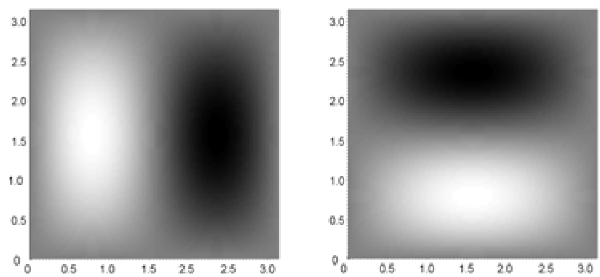

Figure 3.4. Eigenfunctions $u_{1,2}$ and $u_{2,1}$.

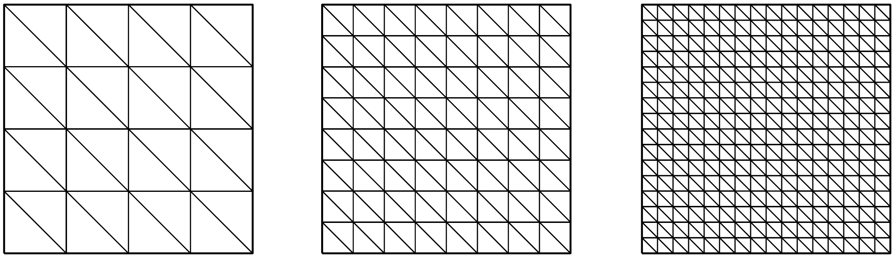

Figure 3.5. Sequence of uniform meshes ($N = 4, 8, 16$).

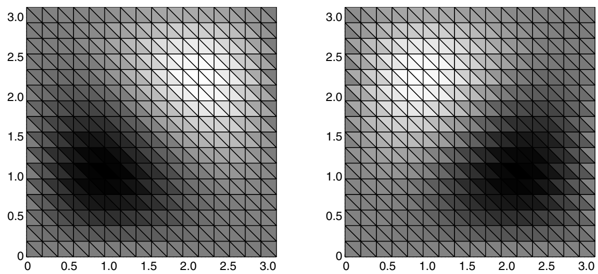

Figure 3.6. Eigenfunctions associated with $\lambda_h^{(2)}$ and $\lambda_h^{(3)}$ on the uniform mesh sequence.

The situation is, however, simpler on a uniform mesh sequence. We consider a mesh sequence of right-angled triangles obtained by bisecting a uniform mesh of squares (see Figure 3.5). Table 3.2 (overleaf) lists the first ten computed eigenvalues and their rate of convergence towards the exact values. This computation does not differ too much from the previous one (besides the fact that the convergence order results are cleaner, since the meshes are now uniform). In particular, the multiple eigenvalues are approximated again by distinct discrete values. The corresponding eigenfunctions are plotted in Figure 3.6 for $N = 16$, where the alignment with the mesh is clearly understood. In order to emphasize the mesh dependence, we performed the same computation on the mesh sequence of Figure 3.7, where the triangles have the opposite orientation from before. The computed eigenvalues are exactly the same as in Table 3.2 (in particular, two distinct eigenvalues approximate $\lambda = 5$) and the eigenfunctions corresponding to the multiple eigenvalue are plotted in Figure 3.8. It is evident that the behaviour has changed due to the change in the orientation of the mesh. This result is not surprising since the problem is invariant under the change of variable induced by the symmetry about the line $y = \pi - x$.

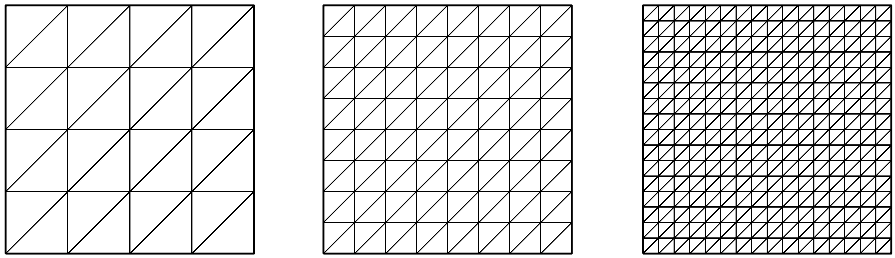

Figure 3.7. Sequence of uniform meshes with reverse orientation ($N = 4, 8, 16$).

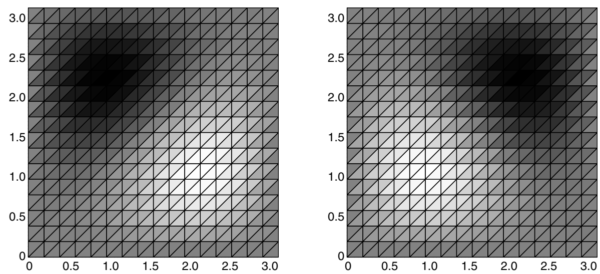

Figure 3.8. Eigenfunctions associated with $\lambda_h^{(2)}$ and $\lambda_h^{(3)}$ on the reversed uniform mesh sequence.

Table 3.2. Eigenvalues computed on the uniform mesh sequence.

Exact	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.3168	2.0776 (2.0)	2.0193 (2.0)	2.0048 (2.0)	2.0012 (2.0)
5	6.3387	5.3325 (2.0)	5.0829 (2.0)	5.0207 (2.0)	5.0052 (2.0)
5	7.2502	5.5325 (2.1)	5.1302 (2.0)	5.0324 (2.0)	5.0081 (2.0)
8	12.2145	9.1826 (1.8)	8.3054 (2.0)	8.0769 (2.0)	8.0193 (2.0)
10	15.5629	11.5492 (1.8)	10.3814 (2.0)	10.0949 (2.0)	10.0237 (2.0)
10	16.7643	11.6879 (2.0)	10.3900 (2.1)	10.0955 (2.0)	10.0237 (2.0)
13	20.8965	15.2270 (1.8)	13.5716 (2.0)	13.1443 (2.0)	13.0362 (2.0)
13	26.0989	17.0125 (1.7)	13.9825 (2.0)	13.2432 (2.0)	13.0606 (2.0)
17	32.4184	21.3374 (1.8)	18.0416 (2.1)	17.2562 (2.0)	17.0638 (2.0)
17		21.5751	18.0705 (2.1)	17.2626 (2.0)	17.0653 (2.0)
DOF	9	49	225	961	3969

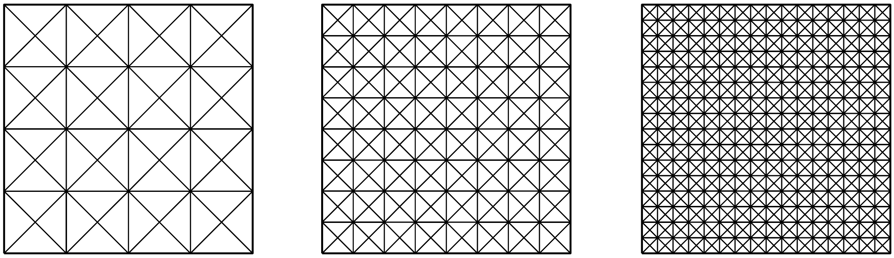

Figure 3.9. Sequence of uniform and symmetric meshes ($N = 4, 8, 16$).

Table 3.3. Eigenvalues computed on the criss-cross mesh sequence.

Exact	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.0880	2.0216 (2.0)	2.0054 (2.0)	2.0013 (2.0)	2.0003 (2.0)
5	5.6811	5.1651 (2.0)	5.0408 (2.0)	5.0102 (2.0)	5.0025 (2.0)
5	5.6811	5.1651 (2.0)	5.0408 (2.0)	5.0102 (2.0)	5.0025 (2.0)
8	9.4962	8.3521 (2.1)	8.0863 (2.0)	8.0215 (2.0)	8.0054 (2.0)
10	12.9691	10.7578 (2.0)	10.1865 (2.0)	10.0464 (2.0)	10.0116 (2.0)
10	12.9691	10.7578 (2.0)	10.1865 (2.0)	10.0464 (2.0)	10.0116 (2.0)
13	17.1879	14.0237 (2.0)	13.2489 (2.0)	13.0617 (2.0)	13.0154 (2.0)
13	17.1879	14.0237 (2.0)	13.2489 (2.0)	13.0617 (2.0)	13.0154 (2.0)
17	25.1471	19.3348 (1.8)	17.5733 (2.0)	17.1423 (2.0)	17.0355 (2.0)
17	38.9073	19.3348 (3.2)	17.5733 (2.0)	17.1423 (2.0)	17.0355 (2.0)
18	38.9073	19.8363 (3.5)	18.4405 (2.1)	18.1089 (2.0)	18.0271 (2.0)
20	38.9073	22.7243 (2.8)	20.6603 (2.0)	20.1634 (2.0)	20.0407 (2.0)
20	38.9073	22.7243 (2.8)	20.6603 (2.0)	20.1634 (2.0)	20.0407 (2.0)
25	38.9073	28.7526 (1.9)	25.8940 (2.1)	25.2201 (2.0)	25.0548 (2.0)
25	38.9073	28.7526 (1.9)	25.8940 (2.1)	25.2201 (2.0)	25.0548 (2.0)
DOF	25	113	481	1985	8065

Our last computation is performed on a uniform and symmetric mesh sequence: the criss-cross mesh sequence of Figure 3.9. The results of this computation are shown in Table 3.3. In this case the multiple eigenvalue $\lambda = 5$ is approximated by pairs of coinciding values. The same happens for the double eigenvalues $\lambda = 10$ (modes $(1,3)$ and $(3,1)$) and $\lambda = 13$ (modes $(2,3)$ and $(3,2)$), while the situation seems different for $\lambda = 17$ (modes $(1,4)$ and $(4,1)$) in the case of the coarsest mesh $N = 4$. This

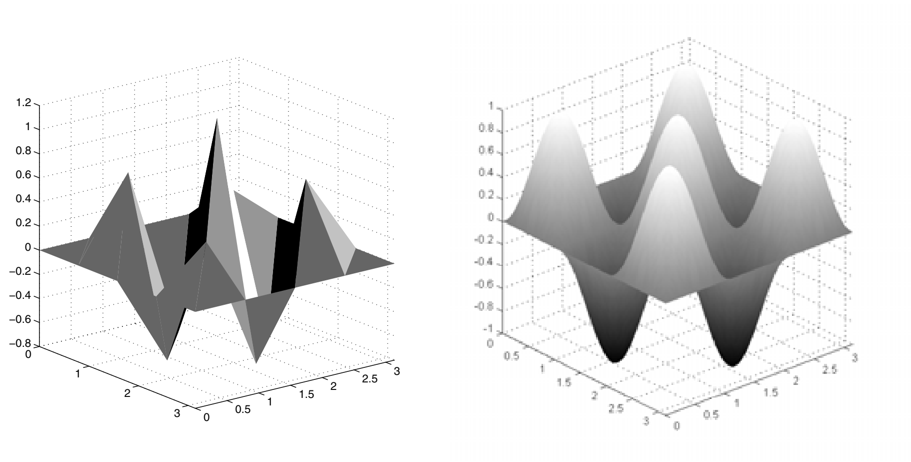

Figure 3.10. Discrete eigenfunction associated to $\lambda_h^{(9)}$ and the exact eigenfunction associated to $\lambda^{(11)}$.

behaviour can be explained as follows: the discrete value $\lambda_h^{(9)} = 25.1471$ is indeed a (bad) approximation of the higher frequency $\lambda^{(11)} = 18$ (mode $(3,3)$). A demonstration of this fact is given by Figure 3.10, which shows the discrete eigenfunction associated to $\lambda_h^{(9)}$ and the exact eigenfunction associated to $\lambda^{(11)}$.

When h is not small enough, we cannot actually expect the order of the discrete eigenvalues to be in a one-to-one correspondence with the continuous ones. For this reason, we include in Table 3.3 five more eigenvalues, which should make the picture clearer.

3.2. The Laplace eigenvalue problem on an L-shaped domain

In all the examples presented so far, the eigenfunctions have been C^∞-functions (they were indeed analytic). We recall here a fundamental example which shows the behaviour of eigenvalue problem approximation when the solution is not smooth.

We consider a domain with a re-entrant corner and the sequence of unstructured triangular meshes shown in Figure 3.11. The shape of our domain is actually a flipped L (the coordinates of the vertices are $(0,0)$, $(1,0)$, $(1,1)$, $(-1,1)$, $(-1,-1)$, and $(0,-1)$), since we use as reference solutions the values proposed in Dauge (2003) where this domain has been considered. In order to compare with the results in the literature, we compute the eigenvalues for the Neumann problem,

$$\begin{aligned} -\Delta u(x,y) &= \lambda u(x,y) && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

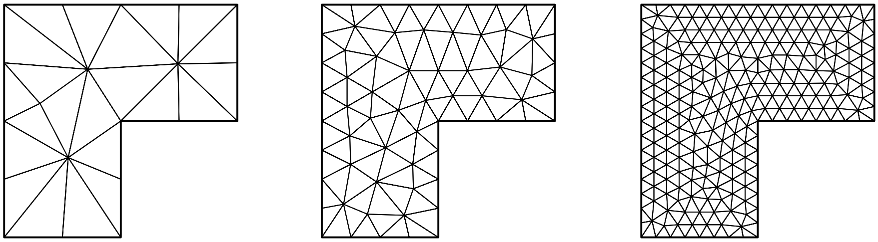

Figure 3.11. Sequence of unstructured mesh for the L-shaped domain ($N = 4, 8, 16$).

Table 3.4. Eigenvalues computed on the L-shaped domain (unstructured mesh sequence).

Exact	Computed (rate)				
	$N=4$	$N=8$	$N=16$	$N=32$	$N=64$
0	−0.0000	0.0000	−0.0000	−0.0000	−0.0000
1.48	1.6786	1.5311 (1.9)	1.4946 (1.5)	1.4827 (1.4)	1.4783 (1.4)
3.53	3.8050	3.5904 (2.3)	3.5472 (2.1)	3.5373 (2.0)	3.5348 (2.0)
9.87	12.2108	10.2773 (2.5)	9.9692 (2.0)	9.8935 (2.1)	9.8755 (2.0)
9.87	12.5089	10.3264 (2.5)	9.9823 (2.0)	9.8979 (2.0)	9.8767 (2.0)
11.39	13.9526	12.0175 (2.0)	11.5303 (2.2)	11.4233 (2.1)	11.3976 (2.1)
DOF	20	65	245	922	3626

using the following variational formulation: find $\lambda \in \mathbb{R}$ and $u \in V$, with $u \neq 0$, such that

$$\int_\Omega \mathbf{grad}\, u(x,y) \cdot \mathbf{grad}\, v(x,)\, \mathrm{d}x\, \mathrm{d}y = \lambda \int_\Omega u(x,y)v(x,y)\, \mathrm{d}x\, \mathrm{d}y \quad \forall v \in V,$$

with $V = H^1(\Omega)$.

The results of the numerical computations are shown in Table 3.4, where we can observe the typical lower approximation rate in the presence of singularities: the first eigenvalue is associated to an eigenspace of singular eigenfunctions, so that the convergence rate deteriorates; on the other hand, the other presented eigenvalues are associated to eigenspaces of smooth functions (since the domain is symmetric), and their convergence is quadratic.

As in the previous examples, we observe that all discrete eigenvalues approximate the continuous ones from above, *i.e.*, we have immediate upper bounds for the exact frequencies.

Since we are considering the Neumann problem, there is a vanishing frequency. Its approximation is zero up to machine precision. In Table 3.4 we display the computed values, rounded to four decimal places, and in some occurrences the zero frequencies turn out to be negative.

Remark 3.1. We have chosen not to refine the mesh in the vicinity of the re-entrant corner, since we wanted to emphasize that the convergence rate of the eigenvalues is related to the smoothness of the corresponding eigenfunction. The convergence in the case of singular solutions can be improved by adding more degrees of freedom where they are needed, but this issue is outside the aim of this work.

3.3. The Laplace eigenvalue problem on the square: non-conforming elements

The last scheme we consider for the approximation of the problem discussed in this section is the linear non-conforming triangular element, also known as the Crouzeix–Raviart method. It is clear that there is an intrinsic interest in studying non-conforming elements; moreover, the approximation of mixed problems (which will be the object of the next examples and will constitute an important part of this survey) can be considered as a sort of non-conforming approximation.

We consider the square domain $\Omega =]0,\pi[\times]0,\pi[$ and compute the eigenvalues on the sequence of unstructured meshes presented in Figure 3.1. The computed frequencies are shown in Table 3.5. As expected, we observe an optimal quadratic convergence.

An important difference with respect to the previous computations is that now all discrete frequencies are lower bounds for the exact solutions. In this particular example all eigenvalues are approximated from below. This is typical behaviour for non-conforming approximation and has been reported by several authors. There is an active literature (see Rannacher (1979) and Armentano and Durán (2004), for instance) on predicting whether non-standard finite element schemes provide upper or lower bounds for eigenvalues, but to our knowledge the question has not yet been answered definitively. Numerical results tend to show that the Crouzeix–Raviart method gives values that are below the exact solutions, but so far only partial results are available.

The general theory we are going to present says that *conforming* approximations of eigenvalues are always *above* the exact solutions, while non-conforming ones may be below. In the mixed approximations shown in the next section there are situations where the same computation provides upper bounds for some eigenvalues and lower bounds for others.

Table 3.5. Eigenvalues computed with the Crouzeix–Raviart method (unstructured mesh sequence).

Exact	Computed (rate)				
	$N=4$	$N=8$	$N=16$	$N=32$	$N=64$
2	1.9674	1.9850 (1.1)	1.9966 (2.1)	1.9992 (2.0)	1.9998 (2.0)
5	4.4508	4.9127 (2.7)	4.9787 (2.0)	4.9949 (2.1)	4.9987 (2.0)
5	4.7270	4.9159 (1.7)	4.9790 (2.0)	4.9949 (2.0)	4.9987 (2.0)
8	7.2367	7.7958 (1.9)	7.9434 (1.9)	7.9870 (2.1)	7.9967 (2.0)
10	8.5792	9.6553 (2.0)	9.9125 (2.0)	9.9792 (2.1)	9.9949 (2.0)
10	9.0237	9.6663 (1.5)	9.9197 (2.1)	9.9796 (2.0)	9.9950 (2.0)
13	9.8284	12.4011 (2.4)	12.8534 (2.0)	12.9654 (2.1)	12.9914 (2.0)
13	9.9107	12.4637 (2.5)	12.8561 (1.9)	12.9655 (2.1)	12.9914 (2.0)
17	10.4013	15.9559 (2.7)	16.7485 (2.1)	16.9407 (2.1)	16.9853 (2.0)
17	11.2153	16.0012 (2.5)	16.7618 (2.1)	16.9409 (2.0)	16.9854 (2.0)
DOF	40	197	832	3443	13972

4. The Laplace eigenvalue problem in mixed form

In this section we present examples which, although classical, are probably not widely known, and which sometimes show a substantially different behaviour from the previous examples.

4.1. The mixed Laplace eigenvalue problem in one dimension

It is classical to rewrite the Laplace problem (2.1) as a first-order system: given $\Omega =]0,\pi[$, find eigenvalues λ and eigenfunctions u with $u \neq 0$, such that, for some s,

$$s(x) - u'(x) = 0 \qquad \text{in } \Omega, \tag{4.1a}$$

$$s'(x) = -\lambda u(x) \qquad \text{in } \Omega, \tag{4.1b}$$

$$u(0) = u(\pi) = 0. \tag{4.1c}$$

Remark 4.1. There are two functions involved with problem (4.1): s and u. In the formulation of the problem, we made explicit that the eigenfunctions we are interested in are the ones represented by u. This might seem a useless remark, since of course in problem (4.1), given u, it turns out that s is uniquely determined as its derivative, and analogously u can be uniquely determined from s and the boundary conditions. On the other hand, this might no longer be true for the discrete case (where the counterpart of our problem will be a degenerate algebraic generalized eigenvalue problem). In particular, we want to define the multiplicity of λ as the dimension of the space associated to the solution u; in general it might turn out that there is more than one s associated with u, and we do not want to consider the multiplicity of s when evaluating the multiplicity of λ.

Given $\Sigma = H^1(\Omega)$ and $U = L^2(\Omega)$, a variational formulation of the mixed problem (4.1) reads as follows: find $\lambda \in \mathbb{R}$ and $u \in U$ with $u \neq 0$, such that, for some $s \in \Sigma$,

$$\int_0^\pi s(x)t(x)\,\mathrm{d}x + \int_0^\pi u(x)t'(x)\,\mathrm{d}x = 0 \quad \forall t \in \Sigma,$$
$$\int_0^\pi s'(x)v(x) = -\lambda \int_0^\pi u(x)v(x)\,\mathrm{d}x \quad \forall v \in U.$$

Its Galerkin discretization is based on discrete subspaces $\Sigma_h \subset \Sigma$ and $U_h \subset U$ and reads as follows: find $\lambda_h \in \mathbb{R}$ and $u_h \in U_h$ with $u_h \neq 0$, such that, for some $s_h \in \Sigma_h$,

$$\int_0^\pi s_h(x)t(x)\,\mathrm{d}x + \int_0^\pi u_h(x)t'(x)\,\mathrm{d}x = 0 \quad \forall t \in \Sigma_h, \tag{4.2a}$$
$$\int_0^\pi s_h'(x)v(x) = -\lambda_h \int_0^\pi u_h(x)v(x)\,\mathrm{d}x \quad \forall v \in U_h. \tag{4.2b}$$

If $\Sigma_h = \mathrm{span}\{\varphi_1, \dots, \varphi_{N_s}\}$ and $U_h = \mathrm{span}\{\psi_1, \dots \psi_{N_u}\}$, then we can introduce the matrices $A = \{a_{kl}\}_{k,l=1}^{N_s}$, $M_U = \{m_{ij}\}_{i,j=1}^{N_u}$ and $B = \{b_{jk}\}$ ($j = 1, \dots, N_u$, $k = 1, \dots, N_s$) as

$$a_{kl} = \int_0^\pi \varphi_l(x)\varphi_k(x)\,\mathrm{d}x,$$
$$m_{ij} = \int_0^\pi \psi_j(x)\psi_i(x)\,\mathrm{d}x,$$
$$b_{jk} = \int_0^\pi \varphi_k'(x)\psi_j(x)\,\mathrm{d}x,$$

so that the algebraic system corresponding to (4.2) has the form

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix} = -\lambda \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix}.$$

4.2. The $\mathcal{P}_1 - \mathcal{P}_0$ element

Given a uniform partition of $[0, \pi]$ of size h, we introduce the most natural lowest-order scheme for the resolution of our problem. Observing that Σ_h and U_h need to approximate $H^1(\Omega)$ and L^2, respectively, and taking advantage of the experience coming from the study of the corresponding source problem (see, for instance, Brezzi and Fortin (1991), Boffi and Lovadina (1997) and Arnold, Falk and Winther (2006*a*)), we use continuous piecewise linear finite elements for Σ_h (that is, conforming $\mathcal{P}_1$ elements) and piecewise constants for U_h (that is, standard $\mathcal{P}_0$). The presented element is actually the one-dimensional counterpart of the well-known lowest-order Raviart–Thomas scheme (see the next section for more details). If N is the number of intervals in our decomposition of Ω, then the involved dimensions are $N_s = N + 1$ and $N_u = N$. In this case it is possible to compute the eigensolutions explicitly. Given that the exact solutions are $\lambda^{(k)} = k^2$ and $u^{(k)}(x) = \sin(kx)$ $(k = 1, 2, \dots)$, we observe that we have $s^{(k)}(x) = k\cos(kx)$. It turns out that the approximate solution for s is its nodal interpolant, that is, $s_h^{(k)}(ih) = k\cos(kih)$, and that the discrete eigenmodes are given by

$$\lambda_h^{(k)} = (6/h^2)\frac{1-\cos kh}{2+\cos kh}, \quad u_h^{(k)}|_{]ih,(i+1)h[} = \frac{s_h^{(k)}(ih) - s_h^{(k)}((i+1)h)}{h\lambda_h^{(k)}},$$

with $k = 1, \dots, N$.

It is quite surprising that the discrete frequencies are exactly the same as in the first example presented in Section 2. There is actually a slight difference in the number of degrees of freedom: here N is the number of intervals, while in Section 2 N was the number of internal nodes, that is, we compute one value more with the mixed scheme on the same mesh. On the other hand, the eigenfunctions are different, as it must be, since here they are piecewise constants while there they were continuous piecewise linears. More precisely, it can be shown that if we consider the exact solution $u^{(k)}(x) = \sin(kx)$, then we have

$$\int_{ih}^{(i+1)h} (u^{(k)}(x) - u_h^{(k)}(x))\,\mathrm{d}x = \frac{\lambda_h - \lambda}{\lambda_h}\int_{ih}^{(i+1)h} u^{(k)}(x)\,\mathrm{d}x.$$

In particular, it turns out that $u_h^{(k)}$ is not the L^2-projection of $u^{(k)}$ onto the piecewise constants space.

4.3. The $\mathcal{P}_1 - \mathcal{P}_1$ element

It is well known that the $\mathcal{P}_1 - \mathcal{P}_1$ element is not stable for the approximation of the one-dimensional Laplace source problem (Babuška and Narasimhan 1997). In particular, it has been shown that it produces acceptable results for smooth solutions, although it is not convergent in the case of singular data. Even though the eigenfunctions of the problem we consider are regular (indeed, they are analytic), the $\mathcal{P}_1 - \mathcal{P}_1$ does not give good results, as we are going to show in this section.

Let us consider again a uniform partition of the interval $[0, \pi]$ into N subintervals and define both Σ_h and U_h as the space of continuous piecewise linear functions (without any boundary conditions). We then have $N_s = N_u = N + 1$.

Table 4.1. Eigenvalues for the one-dimensional mixed Laplacian computed with the $\mathcal{P}_1 - \mathcal{P}_1$ scheme.

Exact	Computed (rate)				
	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
	0.0000	−0.0000	−0.0000	−0.0000	−0.0000
1	1.0001	1.0000 (4.1)	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)
4	3.9660	3.9981 (4.2)	3.9999 (4.0)	4.0000 (4.0)	4.0000 (4.0)
	7.4257	8.5541	8.8854	8.9711	8.9928
9	8.7603	8.9873 (4.2)	8.9992 (4.1)	9.0000 (4.0)	9.0000 (4.0)
16	14.8408	15.9501 (4.5)	15.9971 (4.1)	15.9998 (4.0)	16.0000 (4.0)
25	16.7900	24.5524 (4.2)	24.9780 (4.3)	24.9987 (4.1)	24.9999 (4.0)
	38.7154	29.7390	34.2165	35.5415	35.8846
36	39.0906	35.0393 (1.7)	35.9492 (4.2)	35.9970 (4.1)	35.9998 (4.0)
49		46.7793	48.8925 (4.4)	48.9937 (4.1)	48.9996 (4.0)

Table 4.2. Eigenvalues for the one-dimensional mixed Laplacian computed with the $\mathcal{P}_1 - \mathcal{P}_1$ scheme (the computed values are truncated to ten decimal places).

Exact	Computed (rate)
	$N = 1000$
	−0.0000000000
1	1.0000000000
4	3.9999999999
	8.9998815658
9	8.9999999992
16	15.9999999971
25	24.9999999784
	35.9981051039
36	35.9999999495
49	48.9999998977

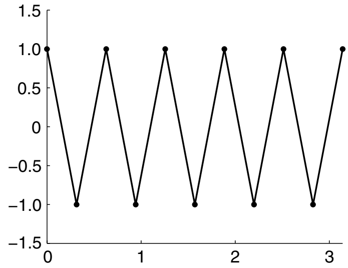

Figure 4.1. Eigenfunction u_h associated to $\lambda_h = 0$.

The results of the numerical computation for increasing N are listed in Table 4.1. The obtained results need some comments. First of all, it is clear that the correct values are well approximated: the rate of convergence is four, meaning that the scheme is of second order (since the rate convergence of the eigenvalues for symmetric eigenproblems, as seen in the previous examples, is doubled). On the other hand, there are some *spurious* solutions which we now describe in more detail.

The zero discrete frequency is related to the fact that the scheme does not satisfy the inf-sup condition. The corresponding eigenfunctions are $s_h(x) \equiv 0$ and $u_h(x)$, as represented in Figure 4.1 in the case $N = 10$. The function u_h is orthogonal in $L^2(0, \pi)$ to all derivatives of functions in Σ_h, and the existence of u_h in this case shows, in particular, that this scheme does not satisfy the classical inf-sup condition. We remark that $\lambda_h = 0$ is a true eigenvalue of our discrete problem even if the corresponding function s_h is vanishing, since the eigenfunction that interests us is u_h (see Remark 4.1).

Besides the zero frequency, there are other spurious solutions: the first one ranges between 7.4257 and 8.9928 in the computations shown in Table 4.1, and is increasing as N increases. Unfortunately, this spurious frequency remains bounded and seems to converge to 9 (which implies the wrong discrete multiplicity for the exact eigenvalue $\lambda = 9$), as is shown in Table 4.2, where we display the results of the computation for $N = 1000$. The same situation occurs for the other spurious value of Tables 4.1 and 4.2, which seems to converge to a value close to 36. The situation is actually more complicated and intriguing: the eigenvalues in the discrete spectrum with rank multiple of four seem spurious, and apparently converge to the value of the next one, that is, $\lambda_h^{(4k)} \to \lambda^{(3k)} = (3k)^2$ for $k = 0, 1, \dots$. The numerically evaluated order of convergence of the spurious frequencies towards $(3k)^2$ is 2. The eigenfunctions corresponding to $\lambda_h^{(4)}$ and $\lambda_h^{(8)}$ are shown in Figure 4.2.

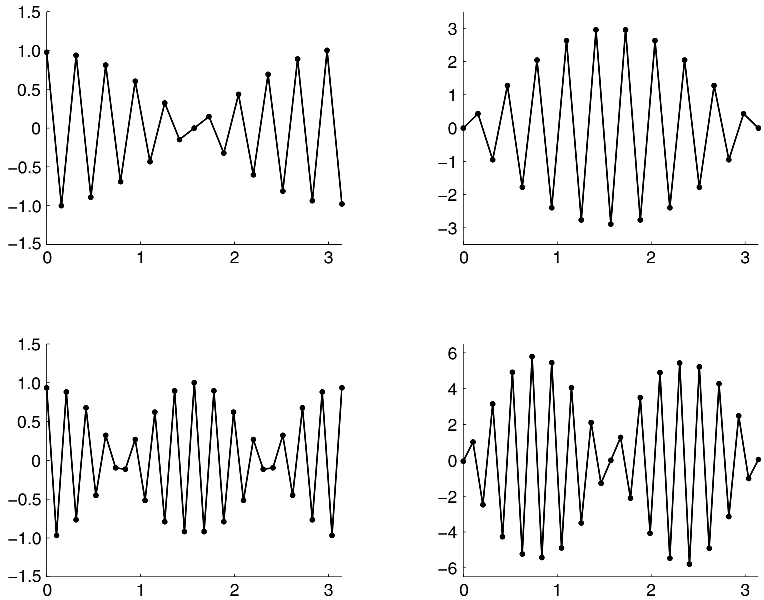

Figure 4.2. $\mathcal{P}_1 - \mathcal{P}_1$ spurious eigenfunctions corresponding to $\lambda_h^{(4)}$ (*above*, $N = 20$) and $\lambda_h^{(8)}$ (*below*, $N = 30$); u_h (*left*) and s_h (*right*).

4.4. The $\mathcal{P}_2 - \mathcal{P}_0$ element

We now discuss briefly the $\mathcal{P}_2 - \mathcal{P}_0$ element, which is known to be unstable for the corresponding source problem (Boffi and Lovadina 1997). The results of the numerical computations on a sequence of successively refined meshes are listed in Table 4.3. In this case there are no spurious solutions, but the computed eigenvalues are wrong by a factor of 6. More precisely, they converge nicely towards six times the exact solutions. The eigenfunctions corresponding to the first two eigenvalues are shown in Figure 4.3: they exhibit behaviour analogous to that observed in the literature for the source problem.

In particular, it turns out that the eigenfunctions u_h are correct approximations of u, while the functions s_h contain spurious components which are clearly associated with a bubble in each element. This behaviour is related to the fact that the ellipticity in the discrete kernel is not satisfied for the presence of the bubble functions in the space $\mathcal{P}_2$. In the case of the source problem, we observed a similar behaviour for s_h, while u_h was a correct approximation of a multiple of u. Here we do not have this phenomenon for u_h since the eigenfunctions are normalized.

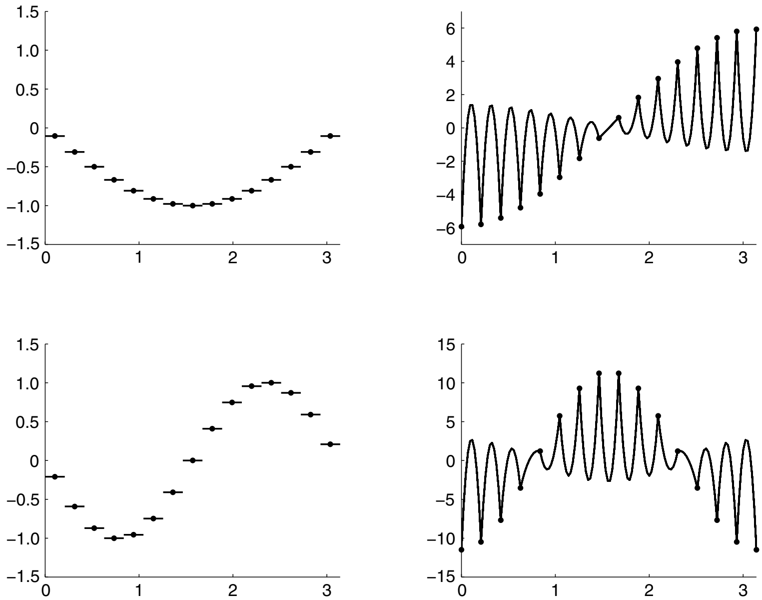

Figure 4.3. $\mathcal{P}_2 - \mathcal{P}_0$ eigenfunctions corresponding to the first (*above*) and the second (*below*) discrete value; u_h (*left*) and s_h (*right*).

Table 4.3. Eigenvalues for the one-dimensional mixed Laplacian computed with the $\mathcal{P}_2 - \mathcal{P}_0$ scheme.

Exact	Computed (rate with respect to 6λ)				
	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
1	5.7061	5.9238 (1.9)	5.9808 (2.0)	5.9952 (2.0)	5.9988 (2.0)
4	19.8800	22.8245 (1.8)	23.6953 (1.9)	23.9231 (2.0)	23.9807 (2.0)
9	36.7065	48.3798 (1.6)	52.4809 (1.9)	53.6123 (2.0)	53.9026 (2.0)
16	51.8764	79.5201 (1.4)	91.2978 (1.8)	94.7814 (1.9)	95.6925 (2.0)
25	63.6140	113.1819 (1.2)	138.8165 (1.7)	147.0451 (1.9)	149.2506 (2.0)
36	71.6666	146.8261 (1.1)	193.5192 (1.6)	209.9235 (1.9)	214.4494 (2.0)
49	76.3051	178.6404 (0.9)	253.8044 (1.5)	282.8515 (1.9)	291.1344 (2.0)
64	77.8147	207.5058 (0.8)	318.0804 (1.4)	365.1912 (1.8)	379.1255 (1.9)
81		232.8461	384.8425 (1.3)	456.2445 (1.8)	478.2172 (1.9)
100		254.4561	452.7277 (1.2)	555.2659 (1.7)	588.1806 (1.9)

4.5. The mixed Laplace eigenvalue problem in two and three space dimensions

Given a domain $\Omega \in \mathbb{R}^n$ $(n = 2, 3)$, the Laplace eigenproblem can be formulated as a first-order system in the following way:

$$\begin{aligned} \boldsymbol{\sigma} - \mathbf{grad}\, u &= 0 && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\sigma} &= -\lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where we introduced the additional variable $\boldsymbol{\sigma} = \mathbf{grad}\, u$. A variational formulation considers the spaces $\Sigma = \mathbf{H}(\operatorname{div};\Omega)$ and $U = L^2(\Omega)$ and reads as follows: find $\lambda \in \mathbb{R}$ and $u \in U$, with $u \neq 0$, such that, for some $\boldsymbol{\sigma} \in \Sigma$,

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \,\mathrm{d}\mathbf{x} + \int_\Omega u \operatorname{div} \boldsymbol{\tau} \,\mathrm{d}\mathbf{x} = 0 \quad \forall \boldsymbol{\tau} \in \Sigma, \tag{4.3a}$$

$$\int_\Omega \operatorname{div} \boldsymbol{\sigma} v \,\mathrm{d}\mathbf{x} = -\lambda \int_\Omega uv \,\mathrm{d}\mathbf{x} \quad \forall v \in U. \tag{4.3b}$$

The Galerkin approximation of our problem consists in choosing finite dimensional subspaces $\Sigma_h \subset \Sigma$ and $U_h \subset U$ and in solving the following discrete problem: find $\lambda_h \in \mathbb{R}$ and $u_h \in U_h$, with $u_h \neq 0$ such that, for some $\boldsymbol{\sigma}_h \in \Sigma_h$,

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} \,\mathrm{d}\mathbf{x} + \int_\Omega u_h \operatorname{div} \boldsymbol{\tau} \,\mathrm{d}\mathbf{x} = 0 \quad \forall \boldsymbol{\tau} \in \Sigma_h, \\ \int_\Omega \operatorname{div} \boldsymbol{\sigma}_h v \,\mathrm{d}\mathbf{x} = -\lambda_h \int_\Omega u_h v \,\mathrm{d}\mathbf{x} \quad \forall v \in U_h. \end{aligned}$$

The algebraic structure of the discrete system is the same as that presented in the one-dimensional case:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix} = -\lambda \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix},$$

where M is a symmetric positive definite matrix.

4.6. Raviart–Thomas elements

We shall use the Raviart–Thomas (RT) elements, which provide the most natural scheme for the approximation of our problem. Similar comments apply to other well-known mixed finite elements, such as Brezzi–Douglas–Marini (BDM) or Brezzi–Douglas–Fortin–Marini (BDFM). We refer the interested reader to Brezzi and Fortin (1991) for a thorough introduction to this subject, and to Raviart and Thomas (1977), Brezzi, Douglas and Marini (1985), and Brezzi, Douglas, Fortin and Marini (1987*b*) for the original definitions.

Table 4.4. Eigenvalues computed with lowest-order RT elements on the uniform mesh sequence of squares.

Exact	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.1048	2.0258 (2.0)	2.0064 (2.0)	2.0016 (2.0)	2.0004 (2.0)
5	5.9158	5.2225 (2.0)	5.0549 (2.0)	5.0137 (2.0)	5.0034 (2.0)
5	5.9158	5.2225 (2.0)	5.0549 (2.0)	5.0137 (2.0)	5.0034 (2.0)
8	9.7268	8.4191 (2.0)	8.1033 (2.0)	8.0257 (2.0)	8.0064 (2.0)
10	13.8955	11.0932 (1.8)	10.2663 (2.0)	10.0660 (2.0)	10.0165 (2.0)
10	13.8955	11.0932 (1.8)	10.2663 (2.0)	10.0660 (2.0)	10.0165 (2.0)
13	17.7065	14.2898 (1.9)	13.3148 (2.0)	13.0781 (2.0)	13.0195 (2.0)
13	17.7065	14.2898 (1.9)	13.3148 (2.0)	13.0781 (2.0)	13.0195 (2.0)
17	20.5061	20.1606 (0.1)	17.8414 (1.9)	17.2075 (2.0)	17.0517 (2.0)
17	20.5061	20.4666 (0.0)	17.8414 (2.0)	17.2075 (2.0)	17.0517 (2.0)
DOF	16	64	256	1024	4096

The RT space is used for the approximation of Σ. One of the main properties is that the finite element space consists of vector fields that are not globally continuous, but only conforming in $\mathbf{H}(\mathrm{div};\Omega)$. This is achieved by requiring the normal component of the vector to be continuous across the elements, and the main tool for achieving this property is the so-called Piola transform, from the reference to the physical element. The space U is approximated by $\mathrm{div}(\Sigma_h)$. In the case of lowest-order elements, in particular, the space U_h is $\mathcal{P}_0$. We refer to Brezzi and Fortin (1991) for more details. We performed the computation on a square domain $\Omega =]0,\pi[^2$ using a sequence of uniform meshes of squares (the parameter N refers to the number of subdivisions of each side). The results of the computations by means of lowest-order RT elements are displayed in Table 4.4. The number of degrees of freedom is evaluated in terms of the variable u_h, since this is the dimension of the algebraic eigenvalue problem to be solved (in this case equal to the number of elements, since u_h is approximated by piecewise constants). From the computed values we can observe that the convergence is quadratic and that all eigenvalues are approximated from above.

The same computation is then performed on a sequence of unstructured triangular meshes such as that presented in Figure 3.1. The results are shown in Table 4.5. In this case the situation is less clear. The theoretical estimates we present again show second order of convergence in h; the reported values, however, even if they are clearly and rapidly converging, are not exactly consistent with the theoretical bound. The reason is that

Table 4.5. Eigenvalues computed with lowest-order RT elements on the unstructured mesh sequence of triangles.

Exact	Computed (rate)				
	$N=4$	$N=8$	$N=16$	$N=32$	$N=64$
2	2.0138	1.9989 (3.6)	1.9997 (1.7)	1.9999 (2.7)	2.0000 (2.8)
5	4.8696	4.9920 (4.0)	5.0000 (8.0)	4.9999 (–2.1)	5.0000 (3.7)
5	4.8868	4.9952 (4.5)	5.0006 (3.0)	5.0000 (5.8)	5.0000 (2.6)
8	8.6905	7.9962 (7.5)	7.9974 (0.6)	7.9995 (2.5)	7.9999 (2.2)
10	9.7590	9.9725 (3.1)	9.9980 (3.8)	9.9992 (1.3)	9.9999 (3.2)
10	11.4906	9.9911 (7.4)	10.0007 (3.7)	10.0005 (0.4)	10.0001 (2.4)
13	11.9051	12.9250 (3.9)	12.9917 (3.2)	12.9998 (5.4)	12.9999 (1.8)
13	12.7210	12.9631 (2.9)	12.9950 (2.9)	13.0000 (7.5)	13.0000 (1.1)
17	13.5604	16.8450 (4.5)	16.9848 (3.4)	16.9992 (4.3)	16.9999 (2.5)
17	14.1813	16.9659 (6.4)	16.9946 (2.7)	17.0009 (2.6)	17.0000 (5.5)
DOF	32	142	576	2338	9400

Table 4.6. Eigenvalues computed with lowest-order RT elements on the uniform mesh sequence of triangles of Figure 3.5.

Exact	Computed (rate)				
	$N=4$	$N=8$	$N=16$	$N=32$	$N=64$
2	2.0324	2.0084 (1.9)	2.0021 (2.0)	2.0005 (2.0)	2.0001 (2.0)
5	4.8340	4.9640 (2.2)	4.9912 (2.0)	4.9978 (2.0)	4.9995 (2.0)
5	5.0962	5.0259 (1.9)	5.0066 (2.0)	5.0017 (2.0)	5.0004 (2.0)
8	8.0766	8.1185 (–0.6)	8.0332 (1.8)	8.0085 (2.0)	8.0021 (2.0)
10	8.9573	9.7979 (2.4)	9.9506 (2.0)	9.9877 (2.0)	9.9969 (2.0)
10	9.4143	9.8148 (1.7)	9.9515 (1.9)	9.9877 (2.0)	9.9969 (2.0)
13	11.1065	12.8960 (4.2)	12.9828 (2.6)	12.9962 (2.2)	12.9991 (2.0)
13	11.3771	13.4216 (1.9)	13.1133 (1.9)	13.0287 (2.0)	13.0072 (2.0)
17	12.2424	16.1534 (2.5)	16.7907 (2.0)	16.9474 (2.0)	16.9868 (2.0)
17	14.7292	16.1963 (1.5)	16.7992 (2.0)	16.9495 (2.0)	16.9874 (2.0)
DOF	32	128	512	2048	8192

RT elements are quite sensitive to the orientation of the mesh. A clean convergence order can be obtained by using a uniform refinement strategy, as in the mesh sequence of Figure 3.5. The results of this computation are listed in Table 4.6

It is interesting to note that in this case the eigenvalues may be approximated from above or below. Even the same eigenvalue can present numerical lower or upper bounds depending on the chosen mesh.

5. The Maxwell eigenvalue problem

Maxwell's eigenvalue problem can be written as follows by means of Ampère and Faraday's laws: given a domain $\Omega \in \mathbb{R}^3$, find the resonance frequencies $\omega \in \mathbb{R}^3$ (with $\omega \neq 0$) and the electromagnetic fields $(\mathbf{E}, \mathbf{H}) \neq (0, 0)$ such that

$$\begin{aligned} \mathbf{curl}\, \mathbf{E} &= \mathrm{i}\omega\mu\mathbf{H} && \text{in } \Omega, \\ \mathbf{curl}\, \mathbf{H} &= -\mathrm{i}\omega\epsilon\mathbf{E} && \text{in } \Omega, \\ \mathbf{E} \times \mathbf{n} &= 0 && \text{on } \partial\Omega, \\ \mathbf{H} \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where we assumed perfectly conducting boundary conditions, and ϵ and μ denote the dielectric permittivity and magnetic permeability, respectively.

From the assumption $\omega \neq 0$ it is well known that we get the usual divergence equations,

$$\begin{aligned} \operatorname{div} \varepsilon\mathbf{E} &= 0 && \text{in } \Omega, \\ \operatorname{div} \mu\mathbf{H} &= 0 && \text{in } \Omega. \end{aligned}$$

For the sake of simplicity, we consider the material properties ϵ and μ constant and equal to the identity matrix. It is outside the scope of this work to consider more general cases; it is remarkable, however, that major mathematical challenges arise even in this simpler situation.

The classical formulation of the eigenvalue problem is obtained from the Maxwell system by eliminating $\mathbf{H}$ (we let $\mathbf{u}$ denote the unknown eigenfunction $\mathbf{E}$): find $\omega \in \mathbb{R}$ and $\mathbf{u} \neq \mathbf{0}$ such that

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} &= \omega^2\mathbf{u} && \text{in } \Omega, && (5.1a) \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, && (5.1b) \\ \mathbf{u} \times \mathbf{n} &= 0 && \text{on } \partial\Omega. && (5.1c) \end{aligned}$$

A standard variational formulation of problem (5.1) reads as follows: find $\omega \in \mathbb{R}$ and $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}; \Omega)$ with $\mathbf{u} \neq \mathbf{0}$ such that

$$\begin{aligned} (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) &= \omega^2(\mathbf{u}, \mathbf{v}) && \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega), && (5.2a) \\ (\mathbf{u}, \mathbf{grad}\,\varphi) &= 0 && \forall \varphi \in H_0^1(\Omega), && (5.2b) \end{aligned}$$

where, as usual, the space $\mathbf{H}_0(\mathbf{curl};\Omega)$ consists of vector fields in $L^2(\Omega)^3$ with $\mathbf{curl}$ in $L^2(\Omega)^3$, and with vanishing tangential trace on the boundary. Here $H^1_0(\Omega)$ is the standard Sobolev space of functions in $L^2(\Omega)$ with $\mathbf{grad}$ in $L^2(\Omega)^3$, and vanishing trace on the boundary.

It is a common practice to consider the following variational formulation for the approximation of problem (5.1): find $\omega \in \mathbb{R}$ and $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ with $\mathbf{u} \neq \mathbf{0}$ such that

$$(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \omega^2(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl};\Omega). \tag{5.3}$$

It is easy to observe that the eigenmodes of (5.3) corresponding to non-vanishing frequencies $\omega \neq 0$ are also solutions to problem (5.2): it is sufficient to choose $\mathbf{v} = \mathbf{grad}\,\varphi$ in (5.3) in order to obtain the second equation of (5.2). When the domain is simply connected, these are the only solutions to problem (5.2): $\omega = 0$ in (5.2) implies $\mathbf{curl}\,\mathbf{u} = \mathbf{0}$ which, together with $\operatorname{div}\mathbf{u} = 0$ and the boundary conditions, means $\mathbf{u} = \mathbf{0}$ if the cohomology is trivial. On the other hand, if there exist non-vanishing vector fields $\mathbf{u}$ with $\mathbf{curl}\,\mathbf{u} = \mathbf{0}$, $\operatorname{div}\mathbf{u} = 0$ in Ω, and $\mathbf{u} \times \mathbf{n}$ on $\partial\Omega$ (harmonic vector fields), then problem (5.2) has solutions with zero frequency $\omega = 0$. These solutions are obviously also present in problem (5.3): in this case the eigenspace corresponding to the zero frequency is made of the harmonic vector fields plus the infinite-dimensional space $\mathbf{grad}(H^1_0(\Omega))$. It is well known that the space of harmonic vector fields is finite-dimensional, its dimension being the first Betti number of Ω.

From now on we assume that Ω is simply connected, and discuss some numerical approximations of the two-dimensional counterpart of problem (5.3).

Following Boffi, Fernandes, Gastaldi and Perugia (1999*b*), it is not difficult to check that problem (5.2) is equivalent to the following: find $\lambda \in \mathbb{R}$ and $\mathbf{p} \in \mathbf{H}_0(\operatorname{div}^0;\Omega)$ with $\mathbf{p} \not\equiv \mathbf{0}$ such that, for some $\boldsymbol{\sigma} \in \mathbf{H}_0(\mathbf{curl};\Omega)$,

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{p}, \mathbf{curl}\,\boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl};\Omega), \tag{5.4a}$$

$$(\mathbf{curl}\,\boldsymbol{\sigma}, \mathbf{q}) = -\lambda(\mathbf{p}, \mathbf{q}) \quad \forall \mathbf{q} \in \mathbf{H}_0(\operatorname{div}^0;\Omega), \tag{5.4b}$$

where $\mathbf{H}_0(\operatorname{div}^0;\Omega)$ denotes the subspace of $\mathbf{H}_0(\operatorname{div};\Omega)$ consisting of divergence-free vector fields and where the equivalence is given by $\lambda = \omega^2$, $\boldsymbol{\sigma} = \mathbf{u}$, and $\mathbf{p} = -\,\mathbf{curl}\,\boldsymbol{\sigma}/\lambda$. The main property used for the proof of equivalence is that $\mathbf{H}_0(\operatorname{div}^0;\Omega)$ coincides with $\mathbf{curl}(\mathbf{H}_0(\mathbf{curl};\Omega))$.

A Galerkin discretization of Maxwell's eigenproblem usually involves a sequence of finite-dimensional subspaces $\Sigma_h \subset \mathbf{H}_0(\mathbf{curl};\Omega)$ so that the approximate formulation reads as follows: find $\omega_h \in \mathbb{R}$ and $\mathbf{u}_h \in \Sigma_h$ with $\mathbf{u}_h \neq \mathbf{0}$ such that

$$(\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}) = \omega_h^2(\mathbf{u}_h, \mathbf{v}) \quad \forall \mathbf{v} \in \Sigma_h. \tag{5.5}$$

The discretization of (5.4) requires two sequences of finite element spaces $\Sigma_h \subset \mathbf{H}_0(\mathbf{curl};\Omega)$ and $U_h \subset \mathbf{H}_0(\mathrm{div}^0;\Omega)$, so that the discrete problem reads as follows: find $\lambda_h \in \mathbb{R}$ and $\mathbf{p}_h \in U_h$ with $\mathbf{p}_h \not\equiv \mathbf{0}$ such that, for some $\boldsymbol{\sigma}_h \in \Sigma_h$,

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\mathbf{p}_h, \mathbf{curl}\,\boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in \Sigma_h, \tag{5.6a}$$

$$(\mathbf{curl}\,\boldsymbol{\sigma}_h, \mathbf{q}) = -\lambda_h(\mathbf{p}_h, \mathbf{q}) \quad \forall \mathbf{q} \in U_h. \tag{5.6b}$$

Boffi *et al.* (1999*b*) showed that, under the assumption

$$\mathbf{curl}(\Sigma_h) = U_h,$$

the same equivalence holds at the discrete level as well: more precisely, all positive frequencies of (5.5) correspond to solutions of (5.6) with the identifications $\lambda_h = \omega_h^2$, $\boldsymbol{\sigma}_h = \mathbf{u}_h$ and $\mathbf{p}_h = -\,\mathbf{curl}\,\boldsymbol{\sigma}_h/\lambda_h$.

Another mixed formulation associated with Maxwell's eigenproblem was introduced in Kikuchi (1987): find $\lambda \in \mathbb{R}$ and $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ with $\mathbf{u} \neq \mathbf{0}$ such that, for some $\psi \in H_0^1(\Omega)$,

$$(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{grad}\,\psi, \mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl};\Omega), \tag{5.7a}$$

$$(\mathbf{u}, \mathbf{grad}\,\varphi) = 0 \quad \forall \varphi \in H_0^1(\Omega). \tag{5.7b}$$

We shall discuss in Section 17 the analogies between the two proposed mixed formulations.

We conclude this preliminary discussion of Maxwell's eigenvalues with a series of two-dimensional numerical results.

5.1. Approximation of Maxwell's eigenvalues on triangular meshes

The two-dimensional counterpart of (5.3) reads as follows: find $\omega \in \mathbb{R}$ and $\mathbf{u} \in \mathbf{H}_0(\mathrm{rot};\Omega)$ with $\mathbf{u} \neq \mathbf{0}$ such that

$$(\mathrm{rot}\,\mathbf{u}, \mathrm{rot}\,\mathbf{v}) = \omega^2(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathrm{rot};\Omega), \tag{5.8}$$

where we used the operator

$$\mathrm{rot}\,\mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = -\,\mathrm{div}(\mathbf{v}^\perp).$$

Its discretization involves a finite-dimensional subspace $\Sigma_h \subset \mathbf{H}_0(\mathrm{rot};\Omega)$ and reads as follows: find $\omega_h \in \mathbb{R}$ and $\mathbf{u}_h \in \Sigma_h$ with $\mathbf{u}_h \neq \mathbf{0}$ such that

$$(\mathrm{rot}\,\mathbf{u}_h, \mathrm{rot}\,\mathbf{v}) = \omega_h^2(\mathbf{u}_h, \mathbf{v}) \quad \forall \mathbf{v} \in \Sigma_h. \tag{5.9}$$

The analogous formulation of (5.4) is as follows: find $\lambda \in \mathbb{R}$ and $p \in \mathrm{rot}(\mathbf{H}_0(\mathrm{rot};\Omega)) = L_0^2(\Omega)$ with $p \not\equiv 0$ such that, for some $\boldsymbol{\sigma} \in \mathbf{H}_0(\mathrm{rot};\Omega)$,

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (p, \mathrm{rot}\,\boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{rot};\Omega), \tag{5.10a}$$

$$(\mathrm{rot}\,\boldsymbol{\sigma}, q) = -\lambda(p, q) \quad \forall q \in L_0^2(\Omega), \tag{5.10b}$$

where $L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ of zero mean-valued functions.

Table 5.1. Eigenvalues computed with lowest-order edge elements on the uniform mesh sequence of triangles of Figure 3.5.

Exact	Computed (rate)				
	$N=4$	$N=8$	$N=16$	$N=32$	$N=64$
1	0.9702	0.9923 (2.0)	0.9981 (2.0)	0.9995 (2.0)	0.9999 (2.0)
1	0.9960	0.9991 (2.2)	0.9998 (2.1)	0.9999 (2.0)	1.0000 (2.0)
2	2.0288	2.0082 (1.8)	2.0021 (2.0)	2.0005 (2.0)	2.0001 (2.0)
4	3.7227	3.9316 (2.0)	3.9829 (2.0)	3.9957 (2.0)	3.9989 (2.0)
4	3.7339	3.9325 (2.0)	3.9829 (2.0)	3.9957 (2.0)	3.9989 (2.0)
5	4.7339	4.9312 (2.0)	4.9826 (2.0)	4.9956 (2.0)	4.9989 (2.0)
5	5.1702	5.0576 (1.6)	5.0151 (1.9)	5.0038 (2.0)	5.0010 (2.0)
8	7.4306	8.1016 (2.5)	8.0322 (1.7)	8.0084 (1.9)	8.0021 (2.0)
9	7.5231	8.6292 (2.0)	8.9061 (2.0)	8.9764 (2.0)	8.9941 (2.0)
9	7.9586	8.6824 (1.7)	8.9211 (2.0)	8.9803 (2.0)	8.9951 (2.0)
zeros	9	49	225	961	3969
DOF	40	176	736	3008	12160

Since the operators rot and div are isomorphic, formulation (5.10) is indeed equivalent to a Neumann problem for the Laplace operator: find $\lambda \in \mathbb{R}$ and $p \in L^2_0(\Omega))$ with $p \not\equiv 0$ such that, for some $\boldsymbol{\sigma} \in \mathbf{H}_0(\mathrm{div};\Omega)$,

$$
\begin{aligned}
(\boldsymbol{\sigma},\boldsymbol{\tau}) + (p, \operatorname{div}\boldsymbol{\tau}) &= 0 \quad \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{div};\Omega),\\
(\operatorname{div}\boldsymbol{\sigma}, q) &= -\lambda(p,q) \quad \forall q \in L^2_0(\Omega)),
\end{aligned}
$$

where the difference with respect to formulation (4.3) is in the boundary conditions, *i.e.*, $\mathbf{H}(\mathrm{div};\Omega)$ is replaced by $\mathbf{H}_0(\mathrm{div};\Omega)$ and, consistently in order to have $\mathrm{div}(\Sigma) = U$, $L^2(\Omega)$ is replaced by $L^2_0(\Omega)$.

For theoretical results on problems analogous to (5.3) and (5.8) involving the divergence operator, we refer the interested reader to Bermúdez and Rodríguez (1994) and Bermúdez *et al.* (1995).

The most natural discretization of problem (5.8) makes use of the so-called *edge* finite elements (Nédélec 1980, 1986). In two space dimensions, edge finite elements are simply standard finite elements used in mixed formulations for the approximation of $\mathbf{H}(\mathrm{div};\Omega)$ (such as RT elements, already seen in the approximation of mixed Laplace eigenvalue problem), rotated by the angle $\pi/2$. The name 'edge finite elements' comes from the nature of the degrees of freedom which, for lowest-order approximation, are associated to moments along the edges of the triangulation. Table 5.1 displays the

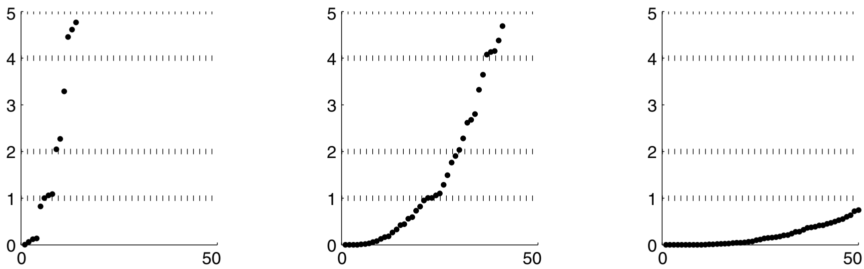

Figure 5.1. First 50 discrete eigenvalues computed with piecewise linears on the unstructured mesh ($N = 4, 8, 16$).

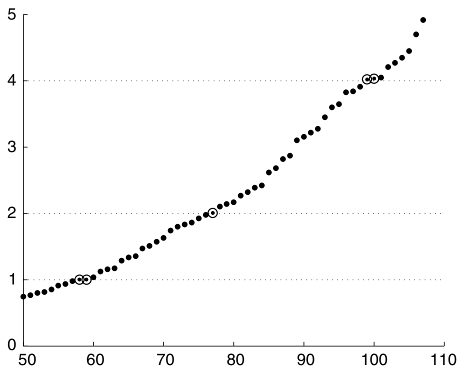

Figure 5.2. Some eigenvalues computed with piecewise linears on the unstructured mesh for $N = 16$.

results of the computation on the square $]0,\pi[^2$ on a sequence of unstructured triangular meshes as in Figure 3.1 with lowest-order edge elements.

Remark 5.1. An important feature of edge element approximation of problem (5.3) is that the zero frequency is approximated by discrete values that are exactly equal to zero (up to machine precision). In the case of lowest-order edge elements, the number of zero frequencies (shown in Table 5.1) is equal to the number of internal vertices of the mesh. This is due to the fact that the elements of Σ_h with vanishing rot coincide with gradients of piecewise linear functions in $H^1_0(\Omega)$.

There have been several attempts to solve problem (5.5) with *nodal* finite elements, that is, standard finite elements in each component with degrees of freedom associated to nodal values. It was soon realized that simulations performed with standard piecewise linears are very sensitive to the used mesh. Figure 5.1 shows the results obtained on the sequence of unstructured triangular meshes of Figure 3.1 with continuous piecewise linear elements in each component. The obtained results can by no means give

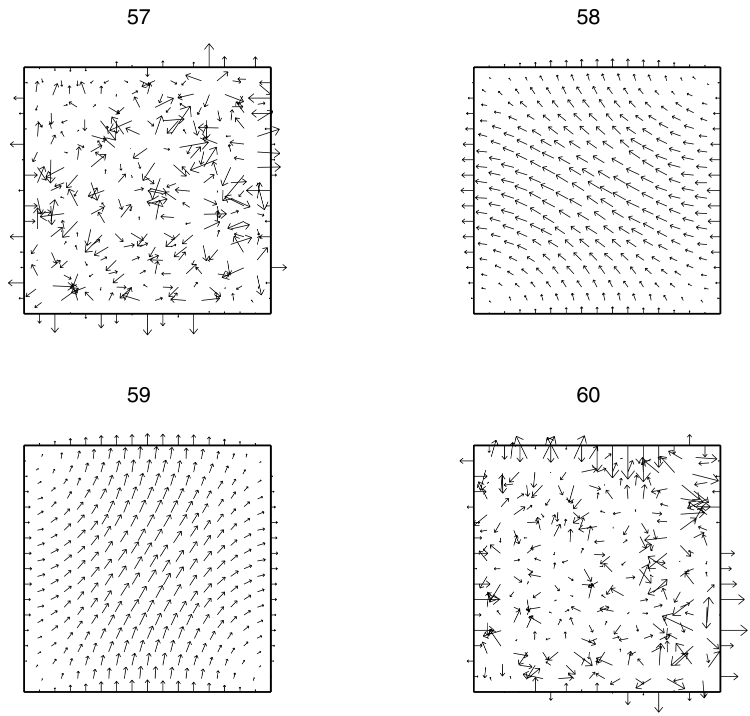

Figure 5.3. Eigenfunctions computed with piecewise linears on the unstructured mesh for $N = 16$.

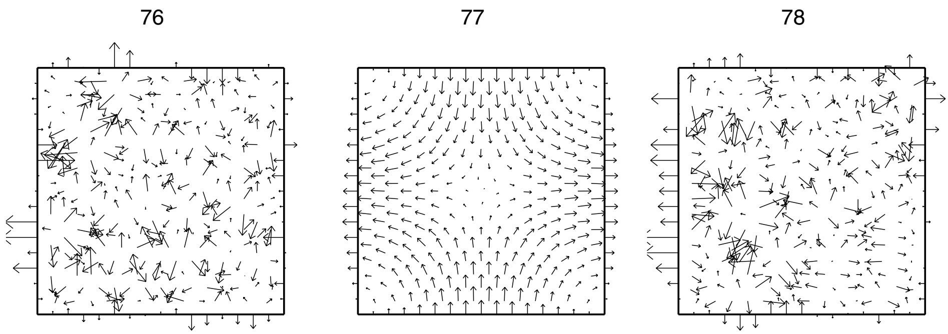

Figure 5.4. More eigenfunctions computed with piecewise linears on the unstructured mesh for $N = 16$.

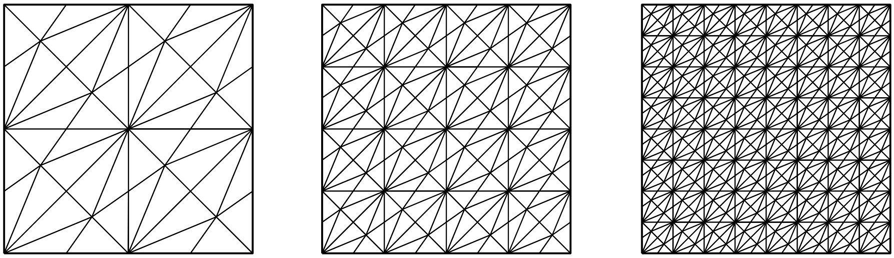

Figure 5.5. Sequence of compatible meshes where gradients are well represented ($N = 2, 4, 8$).

an indication of the exact values. In particular, it is clear that the zero frequency is not well approximated, and it seems that bad approximations of the zero frequency pollute the whole spectrum. Indeed, it can be observed (Boffi *et al.* 1999*b*) that the correct values are well approximated together with their eigenfunctions, but their approximations are hardly distinguishable from the spurious solutions. Figure 5.2 shows the eigenvalues in the range $[50, 110]$ of the spectrum computed with the mesh for $N = 16$. The eigenvalues plotted with different markers are good approximations to the exact solutions. We display some eigenfunctions in Figures 5.3 and 5.4. The different behaviour of the eigenfunctions corresponding to good approximations of the exact solutions and other eigenfunctions can be easily observed.

In view of Remark 5.1, it is clear that a crucial property is that enough gradients are given in the finite element space: this will ensure that the zero frequency is exactly approximated by vanishing discrete eigenvalues. A strategy for designing meshes for which such conditions are satisfied when using piecewise linear elements has been proposed by Wong and Cendes (1988). A sequence of such meshes is plotted in Figure 5.5 and the computed eigenvalues are listed in Table 5.2. It turns out that now several vanishing discrete values correspond to the zero frequency, and that the positive frequencies are optimally approximated.

A rigorous proof of this last statement is not yet available, and for a while there have been researchers who believed that good approximation of the infinite-dimensional kernel was a sufficient condition for the convergence of the eigenmodes. On the other hand, the use of edge elements has to be preferred with respect to nodal elements whenever possible. In order to convince the reader that apparently good results do not necessarily turn out to be correct results, we recall the counter-example presented in Boffi *et al.* (1999*b*). It is actually well known that gradients are well represented by piecewise linears on the criss-cross mesh sequence of Figure 3.9. This is a consequence of results on contour plotting (Powell 1974). The eigenvalues computed with formulation (5.5) using piecewise linears

Table 5.2. Eigenvalues computed with nodal elements on the compatible mesh sequence of triangles of Figure 5.5.

Exact	Computed (rate)				
	$N = 2$	$N = 4$	$N = 8$	$N = 16$	$N = 32$
1	1.0163	1.0045 (1.9)	1.0011 (2.0)	1.0003 (2.0)	1.0001 (2.0)
1	1.0445	1.0113 (2.0)	1.0028 (2.0)	1.0007 (2.0)	1.0002 (2.0)
2	2.0830	2.0300 (1.5)	2.0079 (1.9)	2.0020 (2.0)	2.0005 (2.0)
4	4.2664	4.1212 (1.1)	4.0315 (1.9)	4.0079 (2.0)	4.0020 (2.0)
4	4.2752	4.1224 (1.2)	4.0316 (2.0)	4.0079 (2.0)	4.0020 (2.0)
5	5.2244	5.1094 (1.0)	5.0326 (1.7)	5.0084 (2.0)	5.0021 (2.0)
5	5.5224	5.2373 (1.1)	5.0647 (1.9)	5.0164 (2.0)	5.0041 (2.0)
8	5.8945	8.3376 (2.6)	8.1198 (1.5)	8.0314 (1.9)	8.0079 (2.0)
9	6.3737	9.5272 (2.3)	9.1498 (1.8)	9.0382 (2.0)	9.0096 (2.0)
9	6.8812	9.5911 (1.8)	9.1654 (1.8)	9.0420 (2.0)	9.0105 (2.0)
zeros	7	39	175	735	3007
DOF	46	190	766	3070	12286

(in each component) on the criss-cross mesh sequence are listed in Table 5.3 (page 36). At first glance, the results of the computation might lead to the conclusion that the eigenvalues have been well approximated: the zero frequency is approximated by an increasing number of zero discrete eigenvalues (up to machine precision) and the remaining discrete values are well separated from zero and quadratically converging towards integer numbers. Unfortunately, some limit values do not correspond to exact solutions: *spurious* eigenvalues are computed with this scheme and are indicated with an exclamation mark in Table 5.3. Figure 5.6 shows the eigenfunctions corresponding to the eigenvalues ranging from position 70 to 72 in the spectrum computed with the mesh at level $N = 8$. The checkerboard pattern of the eigenfunction corresponding to eigenvalue number 71, which is the value converging to the spurious solution equal to 6, is evident.

Two more spurious solutions, corresponding to eigenvalues number 79 and 80 (which converge to 15), are displayed in Figure 5.7.

Remark 5.2. All examples presented so far for the approximation of the eigenvalues of Maxwell's equations correspond to standard schemes for the discretization of problem (5.10): find $\lambda_h \in \mathbb{R}$ and $p_h \in \operatorname{rot}(\Sigma_h) = U_h$ with $p_h \not\equiv 0$ such that, for some $\boldsymbol{\sigma}_h \in \Sigma_h$,

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (p_h, \operatorname{rot}\boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in \Sigma_h, \tag{5.11a}$$

$$(\operatorname{rot}\boldsymbol{\sigma}_h, q) = -\lambda_h (p_h, q) \quad \forall q \in U_h. \tag{5.11b}$$

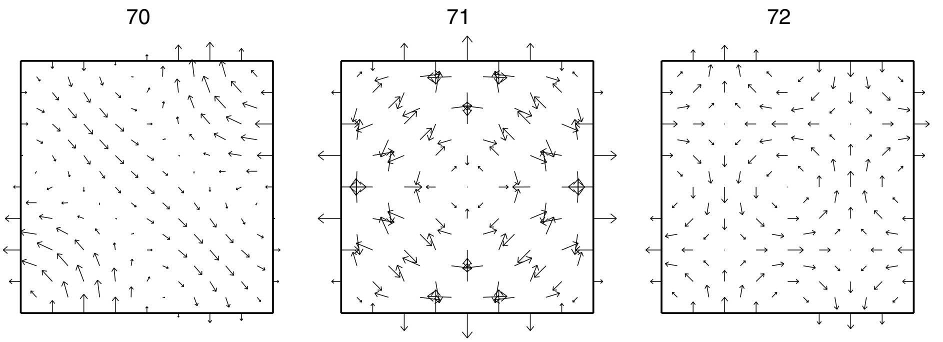

Figure 5.6. The first spurious eigenfunction (*centre*) on the criss-cross mesh for $N = 8$.

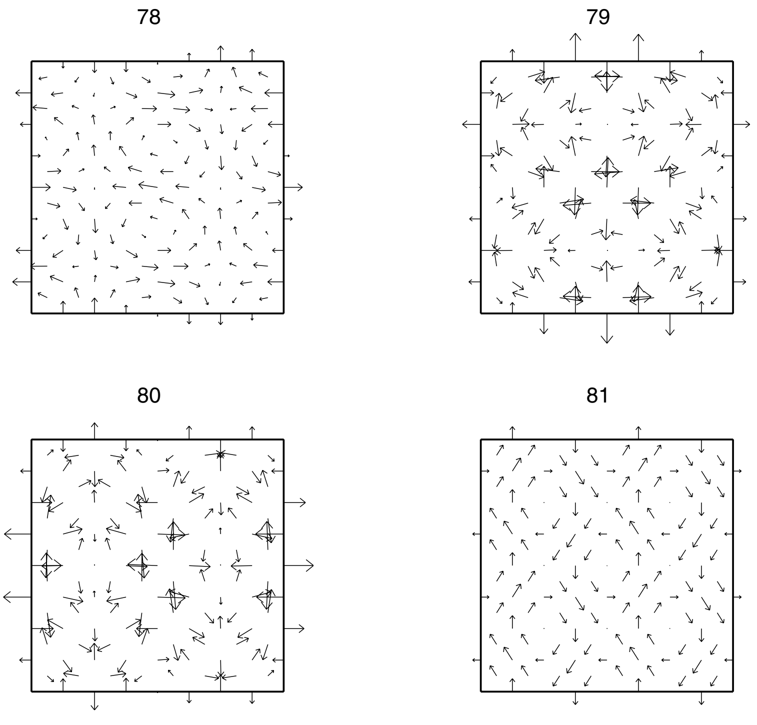

Figure 5.7. The second and third spurious eigenfunctions (numbers 80 and 81) on the criss-cross mesh for $N = 8$.

Table 5.3. Eigenvalues computed with nodal elements on the criss-cross mesh sequence of triangles of Figure 3.9.

Exact	Computed (rate)				
	$N = 2$	$N = 4$	$N = 8$	$N = 16$	$N = 32$
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
2	2.2035	2.0678 (1.6)	2.0171 (2.0)	2.0043 (2.0)	2.0011 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
5	6.1338	5.3971 (1.5)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
5	6.4846	5.3971 (1.9)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
!→ 6	6.4846	5.6712 (0.6)	5.9229 (2.1)	5.9807 (2.0)	5.9952 (2.0)
8	11.0924	8.8141 (1.9)	8.2713 (1.6)	8.0685 (2.0)	8.0171 (2.0)
9	11.0924	10.2540 (0.7)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
9	11.1164	10.2540 (0.8)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
!→15		19.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
!→15		19.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
17		20.9907	18.1813 (1.8)	17.3073 (1.9)	17.0773 (2.0)
zeros	3	15	63	255	1023
DOF	14	62	254	1022	4094

In particular, when Σ_h consists of edge or nodal elements (of lowest order), U_h is the space of piecewise constant functions with zero mean value. All comments made so far then apply to the approximation of the Laplace eigenproblem in mixed form, with the identification discussed above (approximations of the mixed Laplace eigenproblem do not present vanishing discrete values and the eigenfunctions for the formulation in $\mathbf{H}_0(\mathrm{div};\Omega)$ can be obtained from those presented here by rotation of the angle $\pi/2$).

Table 5.4. Eigenvalues computed with edge elements on a sequence of uniform meshes of squares.

Exact	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
2	2.1048	2.0258 (2.0)	2.0064 (2.0)	2.0016 (2.0)	2.0004 (2.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
5	5.9158	5.2225 (2.0)	5.0549 (2.0)	5.0137 (2.0)	5.0034 (2.0)
5	5.9158	5.2225 (2.0)	5.0549 (2.0)	5.0137 (2.0)	5.0034 (2.0)
8	9.7268	8.4191 (2.0)	8.1033 (2.0)	8.0257 (2.0)	8.0064 (2.0)
9	12.8431	10.0803 (1.8)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
9	12.8431	10.0803 (1.8)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
zeros	9	49	225	961	3969
DOF	24	112	480	1984	8064

5.2. Approximation of Maxwell's eigenvalues on quadrilateral meshes

We conclude the discussion of the approximation of Maxwell's eigenvalues with the result of some numerical computations involving quadrilateral meshes.

The first computation, given in Table 5.4, involves edge elements and a sequence of uniform meshes of squares. The discrete eigenvalues converge towards the exact solutions quadratically, as expected, and from above.

In order to warn the reader about possible troubles arising from distorted quadrilateral meshes (in the spirit of the results presented in Arnold, Boffi and Falk (2002, 2005)), in Table 5.5 we present the results of a computation on the sequence of distorted meshes shown in Figure 5.8. In this case the eigenvalues do not converge to the right solution. Indeed, it can be shown that the discrete eigenvalues converge quadratically to incorrect values, which depend on the distortion of the particular mesh used (Gamallo 2002, Bermúdez, Gamallo, Nogueiras and Rodríguez 2006). When using higher-order edge elements, the eigenmodes converge, but with suboptimal rate. Some results on second-order edge elements are reported in Boffi, Kikuchi and Schöberl (2006*c*).

There are several possible cures for this bad behaviour. The first, introduced in Arnold, Boffi and Falk (2005), consists in adding internal degrees of freedom in each element so that the optimal approximation properties

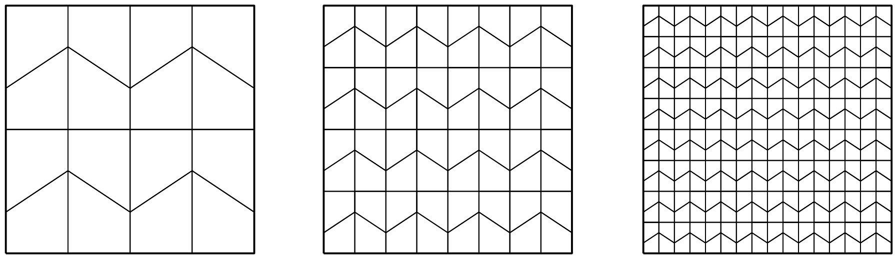

Figure 5.8. Sequence of distorted quadrilateral meshes ($N = 4, 8, 16$).

Table 5.5. Eigenvalues computed with edge elements on the sequence of distorted quadrilaterals of Figure 5.8.

Exact	Computed (rate)				
	$N=4$	$N=8$	$N=16$	$N=32$	$N=64$
1	1.0750	1.0484 (0.6)	1.0418 (0.2)	1.0402 (0.1)	1.0398 (0.0)
1	1.0941	1.0531 (0.8)	1.0430 (0.3)	1.0405 (0.1)	1.0399 (0.0)
2	2.1629	2.1010 (0.7)	2.0847 (0.3)	2.0807 (0.1)	2.0797 (0.0)
4	4.6564	4.3013 (1.1)	4.1936 (0.6)	4.1674 (0.2)	4.1609 (0.1)
4	5.0564	4.3766 (1.5)	4.2124 (0.8)	4.1721 (0.3)	4.1621 (0.1)
5	5.8585	5.3515 (1.3)	5.2362 (0.6)	5.2078 (0.2)	5.2007 (0.0)
5	5.9664	5.4232 (1.2)	5.2539 (0.7)	5.2122 (0.3)	5.2019 (0.1)
8	9.5155	8.6688 (1.2)	8.4046 (0.7)	8.3390 (0.3)	8.3228 (0.1)
9	11.5509	10.0919 (1.2)	9.5358 (1.0)	9.4011 (0.4)	9.3681 (0.1)
9	12.9986	10.4803 (1.4)	9.6307 (1.2)	9.4250 (0.6)	9.3741 (0.2)
zeros	9	49	225	961	3969
DOF	24	112	480	1984	8064

are restored. The convergence analysis for the eigenvalues computed with this new element (sometimes referred to as the ABF element) can be found in Gardini (2005).

Another, cheaper, cure consists in using a projection technique, which can also be interpreted as a reduced integration strategy (Boffi *et al.* 2006*c*). In the lowest-order case it reduces to projecting $\operatorname{rot}\mathbf{u}_h$ onto piecewise constants in formulation (5.9), or, equivalently, to using the midpoint rule in order to evaluate the integral $(\operatorname{rot}\mathbf{u}_h, \operatorname{rot}\mathbf{v})$.

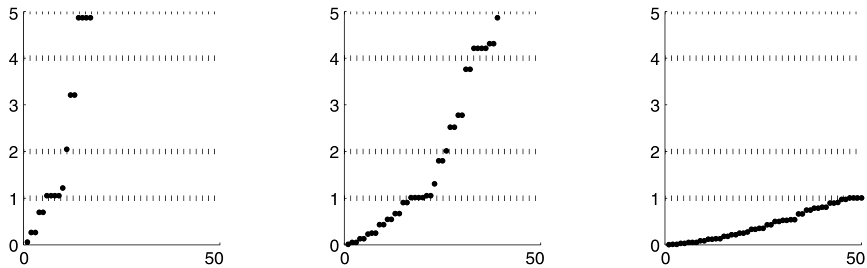

Figure 5.9. First 50 discrete eigenvalues computed with piecewise bilinear elements on the uniform mesh of squares ($N = 4, 8, 16$).

Remark 5.3. On distorted quadrilateral meshes, the equivalence between our original formulation and the mixed formulation (5.11) is no longer true. More precisely, the equivalence is true by choosing $U_h = \mathrm{rot}(\Sigma_h)$, and U_h in this case is not a standard finite element space. For instance, in the case of lowest-order edge elements, U_h in each element K is made of functions $C/|J|$, where $|J|$ is the determinant of the Jacobian of the mapping from the reference cube $\hat{K}$ to K, and C is a generic constant. The projection procedure just described has the effect of changing the discrete formulation, so that in the lowest-order case it turns out to be equivalent to the mixed problem (5.11), with U_h equal to the space of piecewise constant functions. A similar procedure also holds for higher-order edge elements; we refer the interested reader to Boffi *et al.* (2006*c*) for more details.

We conclude this section with some comments on nodal element approximation of Maxwell's eigenvalues on rectangular meshes (Boffi, Durán and Gastaldi 1999*a*). The presented results, in particular, will give some explanations for the spurious eigenvalues shown in Table 5.3.

We start by using classical bilinear elements $\mathcal{Q}_1$ in each component, on a sequence of meshes obtained by dividing the square $\Omega =]0, \pi[$ into N^2 subsquares. The results, which are similar to those obtained by linear elements on unstructured triangular meshes, are given in Figure 5.9. It is clear that these results cannot provide any reasonable approximation to the problem we are interested in.

Another possible scheme consists in projecting $\mathrm{rot}\,\mathbf{u}_h$ onto piecewise constants in formulation (5.9). From the comments made in Remark 5.3, it turns out that this is indeed analogous to considering the $\mathcal{Q}_1 - \mathcal{P}_0$ scheme for mixed Laplacian. The eigenvalues computed with this scheme are given in Table 5.6 and we would like to point out the analogies with the criss-cross computation shown in Table 5.3. It is clear that there is a spurious discrete eigenvalue which converges to 18 (here the term 'spurious' is meant with respect to the multiplicity, since there is in fact an exact solution with value 18 and multiplicity 1). As in the criss-cross example of Table 5.3, there are

Table 5.6. Eigenvalues computed with the projected $\mathcal{Q}_1$ scheme ($\mathcal{Q}_1 - \mathcal{P}_0$) on the sequence of uniform meshes of squares.

Exact	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
2	1.9909	1.9995 (4.1)	2.0000 (4.0)	2.0000 (4.0)	2.0000 (4.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
8	7.2951	7.9636 (4.3)	7.9978 (4.1)	7.9999 (4.0)	8.0000 (4.0)
9	8.7285	10.0803 (–2.0)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
9	11.2850	10.0803 (1.1)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
10	11.2850	10.8308 (0.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
10	12.5059	10.8308 (1.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
13	12.5059	13.1992 (1.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
13	12.8431	13.1992 (–0.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
16	12.8431	14.7608 (1.3)	16.8382 (0.6)	16.2067 (2.0)	16.0515 (2.0)
16		17.5489	16.8382 (0.9)	16.2067 (2.0)	16.0515 (2.0)
17		19.4537	17.1062 (4.5)	17.1814 (–0.8)	17.0452 (2.0)
17		19.4537	17.7329 (1.7)	17.1814 (2.0)	17.0452 (2.0)
!→ 18		19.9601	17.7329 (2.9)	17.7707 (0.2)	17.9423 (2.0)
18		19.9601	17.9749 (6.3)	17.9985 (4.0)	17.9999 (4.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)
zeros	15	63	255	1023	4095
DOF	30	126	510	2046	8190

other spurious solutions with higher frequencies. In this particular case, the closed form of the computed solutions was computed in Boffi *et al.* (1999*a*). It has been shown that the $2(N^2-1)$ degrees of freedom are split into two equal parts: N^2-1 of them correspond to the zero frequency, while the remaining N^2-1 can be ordered in the following way, by means of two indices m,n ranging from 0 to $N-1$ with $m+n\neq 0$:

$$\lambda_h^{(m,n)} = (4/h^2)\frac{\sin^2(\frac{mh}{2})+\sin^2(\frac{nh}{2})-2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1-(2/3)(\sin^2(\frac{mh}{2})+\sin^2(\frac{nh}{2}))+(4/9)\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}.$$

The corresponding eigenfunctions $\mathbf{u}_h^{(m,n)}=(u^{(m,n)},v^{(m,n)})$ are given by

$$u^{(m,n)}(x_i,y_j) = -\frac{2}{h}\sin\left(\frac{mh}{2}\right)\cos\left(\frac{nh}{2}\right)\sin(mx_i)\cos(ny_j), \tag{5.12a}$$

$$v^{(m,n)}(x_i,y_j) = -\frac{2}{h}\cos\left(\frac{mh}{2}\right)\sin\left(\frac{nh}{2}\right)\cos(mx_i)\sin(ny_j). \tag{5.12b}$$

Looking at the formulae (5.12), it seems at first glance that the eigenmodes converge to the exact solution with the correct multiplicity and that there are no spurious solutions. Indeed, given a fixed pair (m,n), it is easy

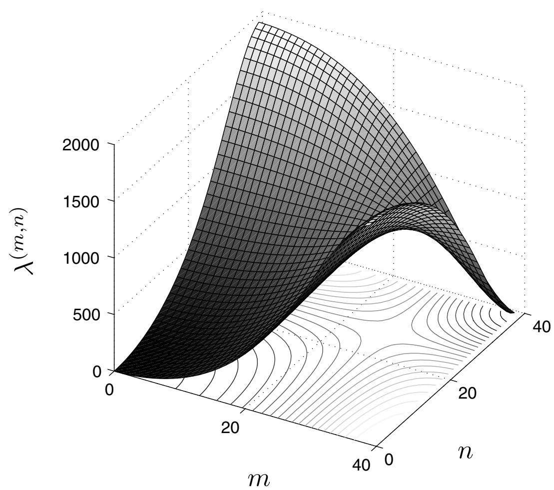

Figure 5.10. Discrete eigenvalues of the $\mathcal{Q}_1-\mathcal{P}_0$ scheme as a function of (m,n).

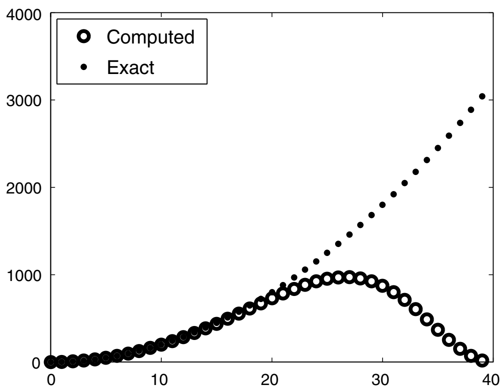

Figure 5.11. Discrete eigenvalues of the $\mathcal{Q}_1 - \mathcal{P}_0$ scheme for $m = n$.

to see that $\lambda_h^{(m,n)}$ tends to $m^2 + n^2$ and $\mathbf{u}_h^{(m.n)}$ to $\mathbf{grad}(\cos(mx)\cos(nx)) = -(m\sin(mx)\cos(nx), n\cos(mx)\sin(nx))$. On the other hand, it can also be easily observed that

$$\lim_{N\to\infty} \lambda_h^{(N-1,N-1)} = 18,$$

where $N = \pi/h$. A clear picture of this phenomenon can be seen in Figure 5.10, where the surface defined by $\lambda_h^{(m,n)}$ is plotted as a function of (m, n). The surface is not convex; in particular, it is not monotone in m and n and, moreover, the value at the corner opposite to the origin tends to 18 as h goes to zero. In Figure 5.11 we also show the section of the surface along the diagonal $m = n$.

With the help of these analytical results, it is possible to sort the eigenvalues of Table 5.6 in a different way, so that the rate of convergence of the spurious eigenvalue can be better evaluated (see Table 5.7).

The behaviour of the presented $\mathcal{Q}_1 - \mathcal{P}_0$ scheme is very similar to that already seen in Table 5.3 for the triangular criss-cross mesh. In that case a closed form of the discrete solution is not available, but can be found for a slight modification of the method (Boffi and Gastaldi 2004).

Remark 5.4. A possible cure for the pathology of the $\mathcal{Q}_1 - \mathcal{P}_0$ scheme was proposed in Chen and Taylor (1990) and analysed for a square domain in Boffi *et al.* (1999*a*). Unfortunately, this method does not seem to provide good results in the case of singular solutions (such as those obtained in an L-shaped domain).

Table 5.7. Eigenvalues computed with the projected $\mathcal{Q}_1$ scheme ($\mathcal{Q}_1 - \mathcal{P}_0$) on the sequence of uniform meshes of squares with the spurious eigenvalue sorted in a different way.

Exact	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
2	1.9909	1.9995 (4.1)	2.0000 (4.0)	2.0000 (4.0)	2.0000 (4.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
8	7.2951	7.9636 (4.3)	7.9978 (4.1)	7.9999 (4.0)	8.0000 (4.0)
9	11.2850	10.0803 (1.1)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
9	11.2850	10.0803 (1.1)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
10	12.5059	10.8308 (1.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
10	12.5059	10.8308 (1.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
13	12.8431	13.1992 (–0.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
13	12.8431	13.1992 (–0.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
16		17.5489	16.8382 (0.9)	16.2067 (2.0)	16.0515 (2.0)
16		19.4537	16.8382 (2.0)	16.2067 (2.0)	16.0515 (2.0)
17		19.4537	17.7329 (1.7)	17.1814 (2.0)	17.0452 (2.0)
17		19.9601	17.7329 (2.0)	17.1814 (2.0)	17.0452 (2.0)
$! \rightarrow$ 18	8.7285	14.7608 (1.5)	17.1062 (1.9)	17.7707 (2.0)	17.9423 (2.0)
18		19.9601	17.9749 (6.3)	17.9985 (4.0)	17.9999 (4.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)
zeros	15	63	255	1023	4095
DOF	30	126	510	2046	8190

PART TWO
Galerkin approximation of compact eigenvalue problems

This part of our survey contains the classical spectral approximation theory for compact operators. It is the core of our work, since all eigenvalue problems we are going to consider are related to compact operators, and we will constantly rely on the fundamental tools described in Section 9. The approximation theory for compact eigenvalue problems has been the object of a wide investigation. A necessarily incomplete list of the most relevant references is Vaĭnikko (1964, 1966), Kato (1966), Anselone and Palmer (1968), Stummel (1970, 1971, 1972), Anselone (1971), Bramble and Osborn (1973), Chatelin (1973), Osborn (1975), Grigorieff (1975*a*, 1975*b*, 1975*c*), Chatelin and Lemordant (1975), Chatelin (1983), Babuška and Osborn (1989, 1991), Kato (1995) and Knyazev and Osborn (2006).

The Galerkin approximation of the Laplace eigenvalue problem, for which we present in Section 8 a rigorous analysis that makes use of standard tools, fits within the framework of the numerical approximation of variationally posed eigenvalue problems, discussed in Sections 7 and 9.

An example of the non-conforming approximation of eigenvalue problems is analysed in Section 11, where a new convergence analysis for Crouzeix–Raviart approximation of the Laplace eigenvalue problem is provided.

6. Spectral theory for compact operators

In this section we present the main definitions we shall need.

Let X be a complex Hilbert space and let $T : X \to X$ be a compact linear operator. The resolvent set $\rho(T)$ is given by the complex numbers $z \in \mathbb{C}$ such that $(zI - T)$ is bijective. We shall use the standard notation $z - T = zI - T$ and the resolvent operator is given by $(z - T)^{-1}$. The spectrum of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$, which is well known to be a countable set with no limit points different from zero. All non-zero values in $\sigma(T)$ are eigenvalues (that is, $z - T$ is one-to-one); zero may or may not be an eigenvalue.

If λ is a non-vanishing eigenvalue of T, then the ascent multiplicity α of $\lambda - T$ is the smallest integer such that $\ker(\lambda - T)^{\alpha} = \ker(\lambda - T)^{\alpha+1}$. The terminology comes from the fact that there also exists a similar definition for the descent multiplicity, which makes use of the range instead of the kernel; for compact operators ascent and descent multiplicities coincide. The dimension of $\ker(\lambda - T)^{\alpha}$ is called the algebraic multiplicity of λ, and the elements of $\ker(\lambda - T)^{\alpha}$ are the generalized eigenvectors of T associated with λ. A generalized eigenvector is of order k if it is in $\ker(\lambda - T)^{k}$, but not in $\ker(\lambda - T)^{k-1}$. The generalized eigenvectors of order 1 are called eigenvectors of T associated with λ, and are the elements of $\ker(\lambda - T)$.

The dimension of $\ker(\lambda - T)$ is called the geometric multiplicity of λ, which is always less than or equal to the algebraic multiplicity. If T is self-adjoint, which will be the case for all examples discussed in this work, then the ascent multiplicity of each eigenvalue is equal to one. This implies that all generalized eigenvectors are eigenvectors, and that the geometric and the algebraic multiplicities coincide.

Given a closed smooth curve $\Gamma \subset \rho(T)$ which encloses $\lambda \in \sigma(T)$, and no other elements of $\sigma(T)$, the Riesz spectral projection associated with λ is defined by

$$E(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (z - T)^{-1} \, \mathrm{d}z. \tag{6.1}$$

The definition clearly does not depend on the chosen curve, and it can be checked that $E(\lambda) : X \to X$, that $E(\lambda) \circ E(\lambda) = E(\lambda)$ (which means it is actually a projection), that $E(\lambda) \circ E(\mu) = 0$ if $\lambda \neq \mu$, that $T \circ E(\lambda) = E(\lambda) \circ T$, and that the range of $E(\lambda)$ is equal to $\ker(\lambda - T)^\alpha$, the space of generalized eigenvectors (which is an invariant subspace for T). This last property will be of fundamental importance for the study of eigenvector approximation, and we emphasize it in the following formula:

$$E(\lambda)X = \ker(\lambda - T)^\alpha.$$

In general, if $\Gamma \subset \rho(T)$ encloses more eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then we have that

$$E(\lambda_1, \lambda_2, \ldots, \lambda_n)X = \bigoplus_{i=1}^{n} \ker(\lambda_i - T)^{\alpha_i},$$

where α_i denotes the ascent multiplicity of $\lambda_i - T$, so that the dimension of the range of the spectral projection is in general the sum of the algebraic multiplicities of the eigenvalues that lie inside Γ.

Let $T^* : X \to X$ denote the adjoint of T. Then $\lambda \in \sigma(T^*)$ if and only if $\overline{\lambda} \in \sigma(T)$, where $\overline{\lambda}$ denotes the conjugate of λ. In particular, the eigenvalues of self-adjoint operators are real. The algebraic multiplicity of $\lambda \in \sigma(T^*)$ is equal to the algebraic multiplicity of $\overline{\lambda} \in \sigma(T)$ and the ascent multiplicity of $\lambda - T^*$ is equal to that of $\overline{\lambda} - T$.

7. Variationally posed eigenvalue problems

In this section we introduce some preliminary results on variationally posed eigenvalue problems. The main theoretical results are presented in Section 9.

The main focus of this survey is on *symmetric* eigenvalue problems, and for this reason we start with symmetric variationally posed eigenvalue problems. Hence, we are dealing with real Hilbert spaces and real-valued bilinear forms. Some discussion of the non-symmetric case can be found in Section 9.

Let V and H be real Hilbert spaces. We suppose that $V \subset H$ with dense and continuous embedding. Let $a : V \times V \to \mathbb{R}$ and $b : H \times H \to \mathbb{R}$

be symmetric and continuous bilinear forms, and consider the following problem: find $\lambda \in \mathbb{R}$ and $u \in V$, with $u \neq 0$, such that

$$a(u, v) = \lambda b(u, v) \quad \forall v \in V. \tag{7.1}$$

We suppose that a is V-elliptic, that is, there exists $\alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|_V^2 \quad \forall v \in V.$$

The ellipticity condition could be weakened by assuming a Gårding-like inequality, that is, $a(\cdot,\cdot) + \mu b(\cdot,\cdot)$ is elliptic for a suitable positive μ. We will not detail this situation, which can be reduced to the elliptic case by a standard shift procedure.

For the sake of simplicity we assume that b defines a scalar product in H. In several applications, H will be $L^2(\Omega)$ and b its standard inner product.

An important tool for the analysis of (7.1) is the solution operator $T : H \to H$: given $f \in H$, our hypotheses guarantee the existence of a unique $Tf \in V$ such that

$$a(Tf, v) = b(f, v) \quad \forall v \in V.$$

Since we are interested in *compact* eigenvalue problems, we make the assumption that

$$T : H \to H \quad \text{is a compact operator,}$$

which is often a consequence of a compact embedding of V into H. We have already observed that we consider T to be self-adjoint.

We assume that the reader is familiar with the spectral theory of compact operators; we recall in particular that the spectrum $\sigma(T)$ of T is a countable or finite set of real numbers with a cluster point possible only at zero. All positive elements of $\sigma(T)$ are eigenvalues with finite multiplicity, and their reciprocals are exactly the eigenvalues of (7.1); moreover, the eigensolutions of (7.1) have the same eigenspaces as those of T.

We let $\lambda^{(k)}$, $k \in \mathbb{N}$, denote the eigenvalues of (7.1) with the natural numbering

$$\lambda^{(1)} \leq \lambda^{(2)} \leq \cdots \leq \lambda^{(k)} \leq \cdots,$$

where the same eigenvalue can be repeated several times according to its multiplicity. We let $u^{(k)}$ denote the corresponding eigenfunctions, with the standard normalization $b(u^{(k)}, u^{(k)}) = 1$, and let $E^{(k)} = \operatorname{span}\{u^{(k)}\}$ denote the associated eigenspaces (see below for multiple eigenvalues). We explicitly observe that even for simple eigenvalues the normalization procedure does not identify $u^{(k)}$ uniquely, but only up to its sign.

It is well known that the eigenfunctions enjoy the orthogonalities

$$a(u^{(m)}, u^{(n)}) = b(u^{(m)}, u^{(n)}) = 0 \quad \text{if } m \neq n, \tag{7.2}$$

which can be deduced easily from (7.1) if $\lambda^{(m)} \neq \lambda^{(n)}$; otherwise, for multiple eigenvalues, when $\lambda^{(m)} = \lambda^{(n)}$, the eigenfunctions $u^{(m)}$ and $u^{(n)}$ can be chosen such that the orthogonalities (7.2) hold.

The Rayleigh quotient is an important tool for the study of eigenvalues: it turns out that

$$\begin{aligned} \lambda^{(1)} &= \min_{v\in V} \frac{a(v,v)}{b(v,v)}, & u^{(1)} &= \arg\min_{v\in V} \frac{a(v,v)}{b(v,v)}, \\ \lambda^{(k)} &= \min_{v\in \left(\bigoplus_{i=1}^{k-1} E^{(i)}\right)^\perp} \frac{a(v,v)}{b(v,v)}, & u^{(k)} &= \arg\min_{v\in \left(\bigoplus_{i=1}^{k-1} E^{(i)}\right)^\perp} \frac{a(v,v)}{b(v,v)}, \end{aligned} \tag{7.3}$$

where it has been implicitly understood (here and in the rest of the paper) that the minima are taken for $v \neq 0$, so that quantities in the denominators do not vanish. The symbol '$\perp$' denotes the orthogonal complement in V with respect to the scalar product induced by the bilinear form b. Due to the orthogonalities (7.2), it turns out that the orthogonal complement could also be taken with respect to the scalar product induced by a.

The Galerkin discretization of problem (7.1) is based on a finite-dimensional space $V_h \subset V$ and reads as follows: find λ_h and $u_h \in V_h$, with $u_h \neq 0$, such that

$$a(u_h, v) = \lambda_h b(u_h, v) \quad \forall v \in V_h. \tag{7.4}$$

Remark 7.1. For historical reasons, we adopt the notation of the h-version of finite elements, and we understand that h is a parameter which tends to zero. Nevertheless, if not explicitly stated, the theory we describe applies to a general Galerkin approximation.

Since V_h is a Hilbert subspace of V, we can repeat the same comments we made for problem (7.1), starting from the definition of the discrete solution operator $T_h : H \to H$: given $f \in H$, $T_h f \in V_h$ is uniquely defined by

$$a(T_h f, v) = b(f, v) \quad \forall v \in V_h.$$

Since V_h is finite-dimensional, T_h is compact; the eigenmodes of T_h are in one-to-one correspondence with those of (7.4) (the equivalence being that the eigenvalues are inverse to each other and the eigenspaces are the same), and we can order the discrete eigenvalues of (7.4) as follows:

$$\lambda_h^{(1)} \leq \lambda_h^{(2)} \leq \cdots \leq \lambda_h^{(k)} \leq \cdots ,$$

where eigenvalues can be repeated according to their multiplicity. We use $u_h^{(k)}$, with the normalization $b(u_h^{(k)}, u_h^{(k)}) = 1$, to denote the discrete eigenfunctions, and $E_h^{(k)} = \operatorname{span}\{u_h^{(k)}\}$ for the associated eigenspaces. Discrete eigenfunctions satisfy the same orthogonalities as the continuous ones,

$$a(u_h^{(m)}, u_h^{(n)}) = b(u_h^{(m)}, u_h^{(n)}) = 0 \quad \text{if } m \neq n,$$

where again this property is a theorem when $\lambda_h^{(m)} \neq \lambda_h^{(n)}$ or a definition when $\lambda_h^{(m)} = \lambda_h^{(n)}$.

Moreover, the properties of the Rayleigh quotient can be applied to discrete eigenmodes as follows:

$$\begin{aligned} \lambda_h^{(1)} &= \min_{v\in V_h} \frac{a(v,v)}{b(v,v)}, & u_h^{(1)} &= \arg\min_{v\in V_h} \frac{a(v,v)}{b(v,v)}, \\ \lambda_h^{(k)} &= \min_{v\in\left(\bigoplus_{i=1}^{k-1} E_h^{(i)}\right)^\perp} \frac{a(v,v)}{b(v,v)}, & u_h^{(k)} &= \arg\min_{v\in\left(\bigoplus_{i=1}^{k-1} E_h^{(i)}\right)^\perp} \frac{a(v,v)}{b(v,v)}, \end{aligned} \tag{7.5}$$

where the symbol '$\perp$' now denotes the orthogonal complement in V_h. An easy consequence of the inclusion $V_h \subset V$ and of the Rayleigh quotient properties is

$$\lambda^{(1)} \le \lambda_h^{(1)},$$

that is, the first discrete eigenvalue is always an upper bound of the first continuous eigenvalue. Unfortunately, equations (7.3) and (7.5) do not allow us to infer any bound between the other eigenvalues, since it is not true in general that $\left(\oplus_{i=1}^{k-1} E_h^{(i)}\right)^\perp$ is a subset of $\left(\oplus_{i=1}^{k-1} E^{(i)}\right)^\perp$. For this reason, we recall the important *min-max* characterization of the eigenvalues.

Proposition 7.2. The kth eigenvalue $\lambda^{(k)}$ of problem (7.1) satisfies

$$\lambda^{(k)} = \min_{E\in V^{(k)}} \max_{v\in E} \frac{a(v,v)}{b(v,v)},$$

where $V^{(k)}$ denotes the set of all subspaces of V with dimension equal to k.

Proof. In order to show that $\lambda^{(k)}$ is greater than or equal to the min-max, take $E = \oplus_{i=1}^k E^{(i)}$, so that $v = \sum_{i=1}^k \alpha_i u^{(i)}$. From the orthogonalities and the normalization of the eigenfunctions, it is easy to obtain the inequality $a(v,v)/b(v,v) \le \lambda^{(k)}$.

The proof of the opposite inequality also gives the additional information that the minimum is attained for $E = \oplus_{i=1}^k E^{(i)}$ and the choice $v = u^{(k)}$. It is clear that if $E = \oplus_{i=1}^k E^{(i)}$ then the optimal choice for v is $u^{(k)}$. On the other hand, if $E \neq \oplus_{i=1}^k E^{(i)}$ then there exists $v \in E$ with v orthogonal to $u^{(i)}$ for all $i \le k$, and hence $a(v,v)/b(v,v) \ge \lambda^{(k)}$, which shows that $E = \oplus_{i=1}^k E^{(i)}$ is the optimal choice for E. □

The analogous min-max condition for the discrete problem (7.4) states

$$\lambda_h^{(k)} = \min_{E_h\in V_h^{(k)}} \max_{v\in E_h} \frac{a(v,v)}{b(v,v)}, \tag{7.6}$$

where $V_h^{(k)}$ denotes the set of all subspaces of V_h with dimension equal to k.

It is then an easy consequence that a conforming approximation $V_h \subset V$ implies that all eigenvalues are approximated from above,

$$\lambda^{(k)} \leq \lambda_h^{(k)} \quad \forall k,$$

since all sets $E_h \in V_h^{(k)}$ in the discrete min-max property are also in $V^{(k)}$, and hence the discrete minimum is evaluated over a smaller set than the continuous one.

Monotonicity is an interesting property (and this behaviour was observed in the numerical examples of Part 1), but it is not enough to show the convergence. The definition of convergence for eigenvalues/eigenfunctions is an intuitive concept, which requires a careful formalism. First of all, we would like the kth discrete eigenvalue to converge towards the kth continuous one. This implies two important facts: all solutions are well approximated and no spurious eigenvalues pollute the spectrum. The numbering we have chosen for the eigenvalues, moreover, implies that the eigenvalues are approximated correctly with their multiplicity. The convergence of eigenfunctions is a little more involved, since we cannot simply require $u_h^{(k)}$ to converge to $u^{(k)}$ in a suitable norm. This type of convergence cannot be expected for at least two good reasons. First of all, the eigenspace associated with multiple eigenvalues can be approximated by the eigenspaces of distinct discrete eigenvalues (see, for example, (3.2)). Then, even in the case of simple eigenvalues, the normalization of the eigenfunctions is not enough to ensure convergence, since they might have the wrong sign. The natural definition of convergence makes use of the notion of the gap between Hilbert spaces, defined by

$$\delta(E, F) = \sup_{\substack{u \in E \\ \|u\|_H = 1}} \inf_{v \in F} \|u - v\|_H,$$

$$\hat{\delta}(E, F) = \max(\delta(E, F), \delta(F, E)).$$

A possible definition of the convergence of eigensolutions was introduced in Boffi, Brezzi and Gastaldi (2000*a*). For every positive integer k, let $m(k)$ denote the dimension of the space spanned by the first distinct k eigenspaces. Then we say that the discrete eigenvalue problem (7.4) converges to the continuous one (7.1) if, for any $\varepsilon > 0$ and $k > 0$, there exists $h_0 > 0$ such that, for all $h < h_0$, we have

$$\begin{aligned} \max_{1 \leq i \leq m(k)} |\lambda^{(i)} - \lambda_h^{(i)}| \leq \varepsilon, \\ \hat{\delta}\left(\bigoplus_{i=1}^{m(k)} E^{(i)}, \bigoplus_{i=1}^{m(k)} E_h^{(i)} \right) \leq \varepsilon. \end{aligned} \tag{7.7}$$

It is remarkable that this definition includes all properties that we need: convergence of eigenvalues and eigenfunctions with correct multiplicity, and absence of spurious solutions.

Remark 7.3. The definition of convergence (7.7) does not give any indication of the approximation rate. It is indeed quite common to separate the convergence analysis for eigenvalue problems into two steps: firstly, the convergence and the absence of spurious modes is investigated in the spirit of (7.7), then suitable approximation rates are proved.

Proposition 7.4. Problem (7.4) converges to (7.1) in the spirit of (7.7) if and only if the following norm convergence holds true:

$$\|T - T_h\|_{\mathcal{L}(H)} \to 0 \quad \text{when } h \to 0. \tag{7.8}$$

Proof. The sufficient part of the proposition is a well-known result in the spectral approximation theory of linear operators (Kato 1995, Chapter IV). The necessity of norm convergence for good spectral approximation of symmetric compact operators was shown in Boffi *et al.* (2000*a*, Theorem 5.1). □

Remark 7.5. Our compactness assumption can be modified by assuming that $T : V \to V$ is compact. In this case norm convergence similar to (7.8) in $\mathcal{L}(V)$ would ensure an analogous eigenmodes convergence of (7.7) with the natural modifications.

In order to show the convergence in norm (7.8), it is useful to recall that the discrete operator T_h can be seen as $T_h = P_h T$, where $P_h : V \to V_h$ is the elliptic projection associated to the bilinear form a. This fact is a standard consequence of Galerkin orthogonality and implies that $T - T_h$ can be written as $(I - P_h)T$, where I denotes the identity operator.

The next proposition can be used to prove convergence in norm.

Proposition 7.6. If T is compact from H to V and P_h converges strongly to the identity operator from V to H, then the norm convergence (7.8) from H to H holds true.

Proof. First we show that the sequence $\{\|I - P_h\|_{\mathcal{L}(V,H)}\}$ is bounded. Define $c(h, u)$ by $\|(I - P_h)u\|_H = c(h, u)\|u\|_V$. Strong (pointwise) convergence means that for each u we have $c(h, u) \to 0$. Thus $M(u) = \max_h c(h, u)$ is finite. By the uniform boundedness principle (or Banach–Steinhaus theorem), there exists C such that, for all h, $\|I - P_h\|_{\mathcal{L}(V,H)} \le C$.

Consider a sequence $\{f_h\}$ such that, for each h, $\|f_h\|_H = 1$ and $\|T - T_h\|_{\mathcal{L}(H)} = \|(T - T_h)f_h\|_H$. Since $\{f_h\}$ is bounded in H and T is compact from H to V, there exists a subsequence, which we again denote by $\{f_h\}$, such that $Tf_h \to w$ in V. We claim that $\|(I - P_h)Tf_h\|_H \to 0$ for the subsequence, and hence for the sequence itself. T is a closed operator: there exists $v \in H$ such that $Tv = w$. By hypothesis $T_h v \to w$; furthermore, $T_h v = P_h T v = P_h w$. The strong convergence of $Tf_h \to w$ in V and

$P_h w \to w$ in H, the triangle inequality and the boundedness of $\{\|I - P_h\|_{\mathcal{L}(V,H)}\}$ imply that, for any $\varepsilon > 0$, there exists h small enough such that

$$\begin{aligned}\|(I-P_h)Tf_h\|_H &\le \|(I-P_h)(Tf_h - w)\|_H + \|(I-P_h)w\|_H \\ &\le C\|Tf_h - w\|_V + \|(I-P_h)w\|_H \le \varepsilon. \qquad \square\end{aligned}$$

Remark 7.7. Results such as that presented in Proposition 7.6 have been used very often in the literature dealing with the approximation theory for linear operators. There are many variants of this, and it is often said that compact operators transform strong (pointwise) into norm (uniform) convergence. It is worth noticing that this result is only true when the compact operator is applied to the right of the converging sequence. For abstract results in this context, we refer, for instance, to Anselone (1971); the same statement as in Proposition 7.6 in the framework of variationally posed eigenvalue problems can be found in Kolata (1978).

Remark 7.8. The same proof as in Proposition 7.6 can be used to show that, if T is compact from V into V, then a stronger pointwise convergence of P_h to the identity, from V into V, is sufficient to ensure the norm convergence

$$\|T - T_h\|_{\mathcal{L}(V)} \to 0 \quad \text{when } h \to 0.$$

Such convergence is equivalent to a type of convergence of eigenvalues and eigenfunctions analogous to (7.7).

8. A direct proof of convergence for Laplace eigenvalues

A fundamental example of elliptic partial differential equation is given by the Laplace operator. Although the convergence theory of the finite element approximation of Laplace eigenmodes is a particular case of the analysis presented in Sections 7 and 9, we now study this basic example. The analysis will be performed with standard tools in the case of Dirichlet boundary conditions and piecewise linear finite elements, but can be applied with minor modifications to Neumann or mixed boundary conditions and to higher-order finite elements. The same technique extends to more general linear eigenvalue problems associated with elliptic operators. Similar arguments can be found in Strang and Fix (1973) and are based on the pioneering work of Birkhoff, de Boor, Swartz and Wendroff (1966) (see also Raviart and Thomas (1983) or Larsson and Thomée (2003)).

Given a polyhedral domain in $\mathbb{R}^3$ (respectively, a polygonal domain in $\mathbb{R}^2$), we are interested in the solution of the following problem: find eigenvalues λ and eigenfunctions u with $u \ne 0$ such that

$$\begin{aligned}-\Delta u &= \lambda u \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega.\end{aligned}$$

A variational formulation of our problem can be obtained by introducing the space $V = H^1_0(\Omega)$ and in looking for $\lambda \in \mathbb{R}$ and $u \in V$, with $u \neq 0$, such that

$$(\mathbf{grad}\, u, \mathbf{grad}\, v) = \lambda(u, v) \quad \forall v \in V. \tag{8.1}$$

Let $V_h \subset V$ be the space of piecewise linear finite elements with vanishing boundary conditions. Then we consider the following approximating problem: find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$, with $u_h \neq 0$, such that

$$(\mathbf{grad}\, u_h, \mathbf{grad}\, v) = \lambda_h(u_h, v) \quad \forall v \in V_h. \tag{8.2}$$

We use the notation of the previous section for the eigensolutions of our continuous and discrete problems. In particular, we adopt the enumeration convention that eigenvalues are repeated according to their multiplicity. We already know from the min-max principle stated in Proposition 7.2 that all eigenvalues are approximated from above, that is,

$$\lambda^{(k)} \leq \lambda_h^{(k)} \quad \forall k,$$

so that, in order to show the convergence of the eigenvalues, we need the upper bound

$$\lambda_h^{(k)} \leq \lambda^{(k)} + \varepsilon(h)$$

with $\varepsilon(h)$ tending to zero as h tends to zero.

We shall use

$$E_h = \Pi_h V^{(k)}$$

in the min-max characterization of the discrete eigenvalues (7.6), where

$$V^{(k)} = \bigoplus_{i=1}^{k} E^{(i)}$$

and $\Pi_h : V \to V_h$ denotes the elliptic projection

$$(\mathbf{grad}(u - \Pi_h u), \mathbf{grad}\, v_h) = 0 \quad \forall v_h \in V_h.$$

In order to do so, we need to check whether the dimension of E_h is equal to k. This can be false in general (for instance, the entire dimension of V_h could be smaller than k), but it is true if h is small enough, as a consequence of the bound

$$\|\Pi_h v\|_{L^2(\Omega)} \geq \|v\|_{L^2(\Omega)} - \|v - \Pi_h v\|_{L^2(\Omega)} \quad \forall v \in V. \tag{8.3}$$

Indeed, if we take h such that

$$\|v - \Pi_h v\|_{L^2(\Omega)} \leq \frac{1}{2}\|v\|_{L^2(\Omega)} \quad \forall v \in V^{(k)}, \tag{8.4}$$

then (8.3) implies that Π_h is injective from $V^{(k)}$ to E_h. It is clear that (8.4) is satisfied for sufficiently small h (how small depending on k).

Taking E_h in the discrete min-max equation (7.6) gives

$$\begin{aligned}\lambda_h^{(k)} &\le \max_{w\in E_h}\frac{\|\mathbf{grad}\, w\|^2_{L^2(\Omega)}}{\|w\|^2_{L^2(\Omega)}} = \max_{v\in V^{(k)}}\frac{\|\mathbf{grad}(\Pi_h v)\|^2_{L^2(\Omega)}}{\|\Pi_h v\|^2_{L^2(\Omega)}}\\ &\le \max_{v\in V^{(k)}}\frac{\|\mathbf{grad}\, v\|^2_{L^2(\Omega)}}{\|\Pi_h v\|^2_{L^2(\Omega)}} = \max_{v\in V^{(k)}}\frac{\|\mathbf{grad}\, v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}}\frac{\|v\|^2_{L^2(\Omega)}}{\|\Pi_h v\|^2_{L^2(\Omega)}}\\ &\le \lambda^{(k)}\max_{v\in V^{(k)}}\frac{\|v\|^2_{L^2(\Omega)}}{\|\Pi_h v\|^2_{L^2(\Omega)}}.\end{aligned}$$

In order to estimate the last term, let us suppose that Ω is convex. Then, it is well known that $V^{(k)}$ is contained in $H^2(\Omega)$ and that

$$\|v-\Pi_h v\|_{L^2(\Omega)} \le Ch^2\|\Delta v\|_{L^2(\Omega)} \le C\lambda^{(k)}h^2\|v\|_{L^2(\Omega)} = C(k)h^2\|v\|_{L^2(\Omega)}.$$

Hence, from (8.3), we obtain

$$\|\Pi_h v\|_{L^2(\Omega)} \ge \|v\|_{L^2(\Omega)}(1-C(k)h^2),$$

which gives the final estimate,

$$\lambda_h^{(k)} \le \lambda^{(k)}\left(\frac{1}{1-C(k)h^2}\right)^2 \simeq \lambda^{(k)}\big(1+C(k)h^2\big)^2 \simeq \lambda^{(k)}(1+2C(k)h^2).$$

In the case of a general domain Ω, it is possible to obtain the following more general result (Raviart and Thomas 1983):

$$\lambda_h^{(k)} \le \lambda^{(k)}\left(1+C(k)\sup_{\substack{v\in V^{(k)}\\ \|v\|=1}}\|v-\Pi_h v\|^2_{H^1(\Omega)}\right). \tag{8.5}$$

We conclude this section with an estimate for the eigenfunctions. It should be clear from the discussion related to estimate (3.2) that the study of the case of multiple eigenvalues is not so simple. For this reason, we start with the situation when $\lambda^{(i)} \ne \lambda^{(k)}$ for all $i \ne k$ (that is, $\lambda^{(k)}$ is a simple eigenvalue).

We introduce the following quantity (Raviart and Thomas 1983):

$$\rho_h^{(k)} = \max_{i\ne k}\frac{\lambda^{(k)}}{|\lambda^{(k)}-\lambda_h^{(i)}|},$$

which makes sense for sufficiently small h since $\lambda^{(k)}$ is a simple eigenvalue and we already know that $\lambda_h^{(i)}$ tends to $\lambda^{(i)} \ne \lambda^{(k)}$.

We also consider the $L^2(\Omega)$-projection of $\Pi_h u^{(k)}$ onto the space spanned by $u_h^{(k)}$,

$$w_h^{(k)} = (\Pi_h u^{(k)}, u_h^{(k)})u_h^{(k)},$$

in order to estimate the difference $(u^{(k)} - u_h^{(k)})$ as follows:

$$\|u^{(k)} - u_h^{(k)}\|_{L^2(\Omega)} \le \|u^{(k)} - \Pi_h u^{(k)}\| + \|\Pi_h u^{(k)} - w_h^{(k)}\| + \|w_h^{(k)} - u_h^{(k)}\|. \quad (8.6)$$

The first term in (8.6) can be easily estimated in terms of powers of h using the properties of Π_h; let us start with the analysis of the second term. From the definition of $w_h^{(k)}$, we have

$$\Pi_h u^{(k)} - w_h^{(k)} = \sum_{i \ne k} (\Pi_h u^{(k)}, u_h^{(i)}) u_h^{(i)},$$

which gives

$$\|\Pi_h u^{(k)} - w_h^{(k)}\|^2 = \sum_{i \ne k} (\Pi_h u^{(k)}, u_h^{(i)})^2. \quad (8.7)$$

We have

$$\begin{aligned}(\Pi_h u^{(k)}, u_h^{(i)}) &= \frac{1}{\lambda_h^{(i)}} (\mathbf{grad}(\Pi_h u^{(k)}), \mathbf{grad}\, u_h^{(i)}) \\ &= \frac{1}{\lambda_h^{(i)}} (\mathbf{grad}\, u^{(k)}, \mathbf{grad}\, u_h^{(i)}) = \frac{\lambda^{(k)}}{\lambda_h^{(i)}} (u^{(k)}, u_h^{(i)}),\end{aligned}$$

that is,

$$\lambda_h^{(i)} (\Pi_h u^{(k)}, u_h^{(i)}) = \lambda^{(k)} (u^{(k)}, u_h^{(i)}).$$

Subtracting $\lambda^{(k)}(\Pi_h u^{(k)}, u_h^{(i)})$ from both sides of the equality, we obtain

$$(\lambda_h^{(i)} - \lambda^{(k)})(\Pi_h u^{(k)}, u_h^{(i)}) = \lambda^{(k)}(u^{(k)} - \Pi_h u^{(k)}, u_h^{(i)}),$$

which gives

$$|(\Pi_h u^{(k)}, u_h^{(i)})| \le \rho_h^{(k)} |(u^{(k)} - \Pi_h u^{(k)}, u_h^{(i)})|.$$

From (8.7) we finally get

$$\begin{aligned}\|\Pi_h u^{(k)} - w_h^{(k)}\|^2 &\le \big(\rho_h^{(k)}\big)^2 \sum_{i \ne k} (u^{(k)} - \Pi_h u^{(k)}), u_h^{(i)})^2 \\ &\le \big(\rho_h^{(k)}\big)^2 \|u^{(k)} - \Pi_h u^{(k)}\|^2.\end{aligned} \quad (8.8)$$

In order to bound the final term in (8.6), we observe that if we show that

$$\|u_h^{(k)} - w_h^{(k)}\| \le \|u^{(k)} - w_h^{(k)}\|, \quad (8.9)$$

then we can conclude that

$$\|u_h^{(k)} - w_h^{(k)}\| \le \|u^{(k)} - \Pi_h u^{(k)}\| + \|\Pi_h u^{(k)} - w_h^{(k)}\|, \quad (8.10)$$

and we have already estimated the last two terms. From the definition of $w_h^{(k)}$, we have

$$u_h^{(k)} - w_h^{(k)} = u_h^{(k)}(1 - ((\Pi_h u^{(k)}, u_h^{(k)})).$$

Moreover,

$$\|u^{(k)}\| - \|u^{(k)} - w_h^{(k)}\| \le \|w_h^{(k)}\| \le \|u^{(k)}\| + \|u^{(k)} - w_h^{(k)}\|,$$

and the normalization of $u^{(k)}$ and $u_h^{(k)}$ gives

$$1 - \|u^{(k)} - w_h^{(k)}\| \le |(\Pi_h u^{(k)}, u_h^{(k)})| \le 1 + \|u^{(k)} - w_h^{(k)}\|,$$

that is,

$$\big||(\Pi_h u^{(k)}, u_h^{(k)})| - 1\big| \le \|u^{(k)} - w_h^{(k)}\|. \tag{8.11}$$

Now comes a crucial point concerning the uniqueness of the normalized eigenfunctions. We have already observed that the normalization of the eigenfunctions does not identify them in a unique way (even in the case of simple eigenvalues), but only up to their sign. Here we have to choose the appropriate sign of $u_h^{(k)}$ in order to have a good approximation of $u^{(k)}$. The correct choice in this case is the one that provides

$$(\Pi_h u^{(k)}, u_h^{(k)}) \ge 0,$$

so that we can conclude that the left-hand side of (8.11) is equal to $\|w_h^{(k)} - u_h^{(k)}\|$ and (8.9) is satisfied.

Putting together all the previous considerations, that is, (8.6), (8.8) and (8.10), we can conclude that, in the case of a simple eigenfunction $u^{(k)}$, there exists an appropriate choice of the sign of $u_h^{(k)}$ such that

$$\|u^{(k)} - u_h^{(k)}\|_{L^2(\Omega)} \le 2(1 + \rho_h^{(k)})\|u^{(k)} - \Pi_h u^{(k)}\|_{L^2(\Omega)}.$$

In particular, in the case of a convex domain this gives the optimal bound

$$\|u^{(k)} - u_h^{(k)}\|_{L^2(\Omega)} \le Ch^2.$$

The error in the energy norm can be estimated in a standard way as follows:

$$\begin{aligned}
C\|u^{(k)} - u_h^{(k)}\|^2_{H^1(\Omega)} &\le \|\mathbf{grad}(u^{(k)} - u_h^{(k)})\|^2_{L^2(\Omega)} \\
&= \|\mathbf{grad}\, u^{(k)}\|^2 - 2(\mathbf{grad}\, u^{(k)}, \mathbf{grad}\, u_h^{(k)}) + \|\mathbf{grad}\, u_h^{(k)}\|^2 \\
&= \lambda^{(k)} - 2\lambda^{(k)}(u^{(k)}, u_h^{(k)}) + \lambda_h^{(k)} \\
&= \lambda^{(k)} - 2\lambda^{(k)}(u^{(k)}, u_h^{(k)}) + \lambda^{(k)} - (\lambda^{(k)} - \lambda_h^{(k)}) \\
&= \lambda^{(k)}\|u^{(k)} - u_h^{(k)}\|^2_{L^2(\Omega)} - (\lambda^{(k)} - \lambda_h^{(k)}).
\end{aligned}$$

The leading term in the last estimate is the second one, which gives the following optimal bound (see (8.5)):

$$\|u^{(k)} - u_h^{(k)}\|_{H^1(\Omega)} \le C(k) \sup_{\substack{v\in V^{(k)}\\ \|v\|=1}} \|v - \Pi_h v\|_{H^1(\Omega)}.$$

In order to conclude the convergence analysis of problem (8.2), it remains to discuss the convergence of eigenfunctions in the case of multiple eigensolutions. As we have already remarked several times, one of the most important issues consists in the appropriate definition of convergence. For the sake of simplicity, we shall discuss the case of a double eigenvalue, but our analysis generalizes easily to any multiplicity. Some of the technical details are identical to the arguments used in the case of an eigenfunction of multiplicity 1, but we repeat them here for the sake of completeness.

Let $\lambda^{(k)}$ be an eigenvalue of multiplicity 2, that is, $\lambda^{(k)} = \lambda^{(k+1)}$ and $\lambda^{(i)} \neq \lambda^{(k)}$ for $i \neq k, k+1$. We would like to find a good approximation for $u^{(k)}$ trying to mimic what has been done in the case of multiplicity 1. Analogous considerations hold for the approximation of $u^{(k+1)}$. It is clear that we cannot expect $u_h^{(k)}$ to converge to $u^{(k)}$, as was observed in the discussion related to (3.2); hence we look for an appropriate linear combination of two discrete eigenfunctions:

$$w_h^{(k)} = \alpha_h u_h^{(k)} + \beta_h u_h^{(k+1)}.$$

From the above study, it seems reasonable to make the following choice:

$$\alpha_h = (\Pi_h u^{(k)}, u_h^{(k)}), \quad \beta_h = (\Pi_h u^{(k)}, u_h^{(k+1)}),$$

so that $w_h^{(k)}$ will be the $L^2(\Omega)$-projection of $\Pi_h u^{(k)}$ onto the space spanned by $u_h^{(k)}$ and $u_h^{(k+1)}$. The analogue of (8.6) then contains two terms,

$$\|u^{(k)} - w_h^{(k)}\|_{L^2(\Omega)} \le \|u^{(k)} - \Pi_h u^{(k)}\| + \|\Pi_h u^{(k)} - w_h^{(k)}\|,$$

and only the last one needs to be estimated. We have

$$\Pi_h u^{(k)} - w_h^{(k)} = \sum_{i\neq k,k+1} (\Pi_h u^{(k)}, u_h^{(i)}) u_h^{(i)},$$

which gives

$$\|\Pi_h u^{(k)} - w_h^{(k)}\|^2 = \sum_{i\neq k,k+1} (\Pi_h u^{(k)}, u_h^{(i)})^2. \tag{8.12}$$

It follows that

$$\begin{aligned}(\Pi_h u^{(k)}, u_h^{(i)}) &= \frac{1}{\lambda_h^{(i)}} (\mathbf{grad}(\Pi_h u^{(k)}), \mathbf{grad}\, u_h^{(i)})\\ &= \frac{1}{\lambda_h^{(i)}} (\mathbf{grad}\, u^{(k)}, \mathbf{grad}\, u_h^{(i)}) = \frac{\lambda^{(k)}}{\lambda_h^{(i)}} (u^{(k)}, u_h^{(i)}),\end{aligned}$$

that is,

$$\lambda_h^{(i)}(\Pi_h u^{(k)}, u_h^{(i)}) = \lambda^{(k)}(u^{(k)}, u_h^{(i)}).$$

Subtracting $\lambda^{(k)}(\Pi_h u^{(k)}, u_h^{(i)})$ from both sides of the equality, we obtain

$$(\lambda_h^{(i)} - \lambda^{(k)})(\Pi_h u^{(k)}, u_h^{(i)}) = \lambda^{(k)}(u^{(k)} - \Pi_h u^{(k)}, u_h^{(i)}),$$

which gives

$$|(\Pi_h u^{(k)}, u_h^{(i)})| \le \rho_h^{(k)} |(u^{(k)} - \Pi_h u^{(k)}, u_h^{(i)})|$$

with the appropriate definition of $\rho_h^{(k)}$,

$$\rho_h^{(k)} = \max_{i\ne k,k+1} \frac{\lambda^{(k)}}{|\lambda^{(k)} - \lambda_h^{(i)}|},$$

which makes sense for sufficiently small h, since we know that $\lambda_h^{(i)}$ tends to $\lambda^{(i)} \ne \lambda^{(k)}$ for $i \ne k, k+1$. From (8.12) we finally get

$$\begin{aligned}\|\Pi_h u^{(k)} - w_h^{(k)}\|^2 &\le \big(\rho_h^{(k)}\big)^2 \sum_{i\ne k,k+1} (u^{(k)} - \Pi_h u^{(k)}), u_h^{(i)})^2 \\ &\le \big(\rho_h^{(k)}\big)^2 \|u^{(k)} - \Pi_h u^{(k)}\|^2,\end{aligned}$$

which gives the optimal bound

$$\|u^{(k)} - w_h^{(k)}\|_{L^2(\Omega)} \le (1 + \rho_h^{(k)})\|u^{(k)} - \Pi_h u^{(k)}\|_{L^2(\Omega)}.$$

The derivation of convergence estimates in the energy norm is less immediate, since we cannot repeat the argument used for the case of eigensolutions of multiplicity 1. The main difference is that the approximating eigenfunction $w_h^{(k)}$ is not normalized. However, the proof can be modified as follows:

$$\begin{aligned}C\|u^{(k)} - w_h^{(k)}\|^2_{H^1(\Omega)} &\le \|\mathbf{grad}(u^{(k)} - w_h^{(k)})\|^2_{L^2(\Omega)} \\ &= \|\mathbf{grad}\, u^{(k)}\|^2 - 2(\mathbf{grad}\, u^{(k)}, \mathbf{grad}\, w_h^{(k)}) + \|\mathbf{grad}\, w_h^{(k)}\|^2 \\ &= \lambda^{(k)} - 2\lambda^{(k)}(u^{(k)}, w_h^{(k)}) + \alpha_h^2\lambda_h^{(k)} + \beta_h^2\lambda_h^{(k+1)} \\ &= \lambda^{(k)} - 2\lambda^{(k)}(u^{(k)}, w_h^{(k)}) + (\alpha_h^2 + \beta_h^2)\lambda^{(k)} \\ &\qquad - ((\alpha_h^2 + \beta_h^2)\lambda^{(k)} - \alpha_h^2\lambda_h^{(k)} - \beta_h^2\lambda_h^{(k+1)}) \\ &= \lambda^{(k)}\|u^{(k)} - w_h^{(k)}\|^2_{L^2(\Omega)} - \alpha_h^2(\lambda^{(k)} - \lambda_h^{(k)}) - \beta_h^2(\lambda^{(k)} - \lambda_h^{(k+1)})\end{aligned}$$

and we get the optimal estimate

$$\|u^{(k)} - w_h^{(k)}\|_{H^1(\Omega)} \le C(k) \sup_{\substack{v\in V^{(k+1)} \\ \|v\|=1}} \|v - \Pi_h v\|_{H^1(\Omega)}.$$

The following theorem summarizes the results obtained so far.

Theorem 8.1. Let $(\lambda^{(i)}, u^{(i)})$ be the solutions of problem (8.1) with the notation of Section 7, and let $(\lambda_h^{(i)}, u_h^{(i)})$ be the corresponding discrete solutions of problem (8.2). Let $\Pi_h : V \to V_h$ denote the elliptic projection. Then, for any k not larger than the dimension of V_h, for sufficiently small h, we have

$$\lambda^{(k)} \le \lambda_h^{(k)} \le \lambda^{(k)} + C(k) \sup_{\substack{v\in V^{(k)}\\ \|v\|=1}} \|v - \Pi_h v\|^2_{H^1(\Omega)},$$

with $V^{(k)} = \oplus_{i\le k} E^{(k)}$.

Moreover, let $\lambda^{(k)}$ be an eigenvalue of multiplicity $m \ge 1$, so that

$$\lambda^{(k)} = \cdots = \lambda^{(k+m-1)} \quad \text{and} \quad \lambda^{(i)} \ne \lambda^{(k)}$$

for $i \ne k, \ldots, k+m-1$. Then there exists

$$\{w_h^{(k)}\} \subset E_h^{(k)} \oplus \cdots \oplus E_h^{(k+m-1)}$$

such that

$$\|u^{(k)} - w_h^{(k)}\|_{H^1(\Omega)} \le C(k) \sup_{\substack{v\in V^{(k+m-1)}\\ \|v\|=1}} \|v - \Pi_h v\|_{H^1(\Omega)}$$

and

$$\|u^{(k)} - w_h^{(k)}\|_{L^2(\Omega)} \le C(k) \|u^{(k)} - \Pi_h u^{(k)}\|_{L^2(\Omega)}.$$

The results presented so far are optimal when all the eigenfunctions are smooth. For instance, if the domain is convex, it is well known that $\|v - \Pi_h v\|_{H^1(\Omega)} = O(h)$ for all eigenfunctions v, so that Theorem 8.1 gives the optimal second order of convergence for the eigenvalues, first order in $H^1(\Omega)$ for the eigenfunctions, and second order in $L^2(\Omega)$ for the eigenfunctions. On the other hand, if the domain is not regular (see, for instance, the computations presented in Section 3.2) it usually turns out that some eigenspaces contain smooth eigenfunctions, while others may contain singular eigenfunctions. In such cases, we obtain from Theorem 8.1 a sub-optimal estimate, since some bounds are given in terms of the approximability of $V^{(k)}$. Hence, if we are interested in the kth eigenvalue, we have to consider the regularity properties of all the eigenspaces up to the kth one. This sub-optimal behaviour is not observed in practice (see, for instance, Table 3.4); the theoretical investigations presented in the subsequent sections will confirm that the rate of convergence of the kth eigenvalue/eigenfunction is indeed related to the approximability of the eigenfunctions associated with the kth eigenvalue, alone. For sharper results concerning multiple eigenvalues, the reader is referred to Knyazev and Osborn (2006).

9. The Babuška–Osborn theory

It is generally understood that the basic reference for the finite element approximation of compact eigenvalue problems is the so-called Babuška–Osborn theory (Babuška and Osborn 1991). In this section, we recall the main results of the theory, and refer the reader to the original reference for more details.

While the main focus of this survey is on symmetric eigenvalue problems, it is more convenient to embed the discussion of the present section in the complex field $\mathbb{C}$ and to study a generic non-symmetric problem. Those interested in wider generality may observe that the original theory can be developed in Banach spaces; we shall, however, limit ourselves to the interesting case of Hilbert spaces.

We follow the notation introduced in Section 6.

Let X be a Hilbert space and let $T : X \to X$ be a compact linear operator. We consider a family of compact operators $T_h : X \to X$ such that

$$\|T - T_h\|_{\mathcal{L}(X)} \to 0 \quad \text{as } h \to 0. \tag{9.1}$$

In our applications, T_h will be a finite rank operator. We have already seen examples of this situation in Section 7.

As a consequence of (9.1), if $\lambda \in \sigma(T)$ is a non-zero eigenvalue with algebraic multiplicity m, then exactly m discrete eigenvalues of T_h (counted with their algebraic multiplicities), converge to λ as h tends to zero. This follows from the well-known fact that, given an arbitrary closed curve $\Gamma \subset \rho(T)$ as in the definition (6.1) of the projection $E(\lambda)$, for sufficiently small h we have $\Gamma \subset \rho(T_h)$, and Γ encloses exactly m eigenvalues of T_h, counted with their algebraic multiplicities. More precisely, for sufficiently small h it makes sense to consider the discrete spectral projection

$$E_h(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (z - T_h)^{-1} \,\mathrm{d}z,$$

and it turns out that the dimension of $E_h(\lambda)X$ is equal to m. Moreover,

$$\|E(\lambda) - E_h(\lambda)\|_{\mathcal{L}(X)} \to 0 \quad \text{as } h \to 0,$$

which implies the convergence of the generalized eigenvectors.

It is common practice, when studying the approximation of eigenmodes, to split the convergence analysis into two parts: the first step consists in showing that the eigenmodes converge and that there are no spurious solutions, the second one deals with the order of convergence.

The above considerations give an answer to the first question and we summarize these results in the following statement.

Theorem 9.1. Let us assume that the convergence in norm (9.1) is satisfied. For any compact set $K \subset \rho(T)$, there exists $h_0 > 0$ such that, for all $h < h_0$, we have $K \subset \rho(T_h)$ (absence of spurious modes). If λ is a non-zero eigenvalue of T with algebraic multiplicity equal to m, then there are m eigenvalues $\lambda_{1,h}, \lambda_{2,h}, \dots, \lambda_{m,h}$ of T_h, repeated according to their algebraic multiplicities, such that each $\lambda_{i,h}$ converges to λ as h tends to 0.

Moreover, the gap between the direct sum of the generalized eigenspaces associated with $\lambda_{1,h}, \lambda_{2,h}, \dots, \lambda_{m,h}$ and the generalized eigenspace associated to λ tends to zero as h tends to 0.

We now report the main results of the Babuška–Osborn theory (Babuška and Osborn 1991, Theorems 7.1–7.4) which deal with the convergence order of eigenvalues and eigenvectors. One of the main applications of the theory consists in the convergence analysis for variationally posed eigenvalue problems (Babuška and Osborn 1991, Theorems 8.1–8.4); this is the correct setting for the general analysis of the problems discussed in Sections 7 and 8. We start with the generalization of the framework of Section 7 to non-symmetric variationally posed eigenvalue problems.

Let V_1 and V_2 be complex Hilbert spaces. We are interested in the following eigenvalue problem: find $\lambda \in \mathbb{C}$ and $u \in V_1$, with $u \neq 0$, such that

$$a(u,v) = \lambda b(u,v) \quad \forall v \in V_2, \tag{9.2}$$

where $a : V_1 \times V_2 \to \mathbb{C}$ and $b : V_1 \times V_2 \to \mathbb{C}$ are sesquilinear forms. The form a is assumed to be continuous,

$$|a(v_1,v_2)| \le C\|v_1\|_{V_1}\|v_2\|_{V_2} \quad \forall v_1 \in V_1 \ \forall v_2 \in V_2,$$

and the form b is continuous with respect to a *compact norm*: there exists a norm $\|\cdot\|_{H_1}$ in V_1 such that any bounded sequence in V_1 has a Cauchy subsequence with respect to $\|\cdot\|_{H_1}$ and

$$|b(v_1,v_2)| \le C\|v_1\|_{H_1}\|v_2\|_{V_2} \quad \forall v_1 \in V_1 \ \forall v_2 \in V_2.$$

The Laplace eigenvalue problem considered in Sections 8 and 10 fits within this setting with the choices $V_1 = V_2 = H^1_0(\Omega)$ and $H_1 = L^2(\Omega)$.

In order to define the solution operators, we assume the inf-sup condition

$$\inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \frac{|a(v_1,v_2)|}{\|v_1\|_{V_1}\|v_2\|_{V_2}} \ge \gamma > 0,$$
$$\sup_{v_1 \in V_1} |a(v_1,v_2)| > 0 \quad \forall v_2 \in V_2 \setminus \{0\},$$

so that we can introduce $T : V_1 \to V_1$ and $T_* : V_2 \to V_2$ by

$$a(Tf,v) = b(f,v) \quad \forall f \in V_1 \ \forall v \in V_2,$$
$$a(v,T_*g) = b(v,g) \quad \forall g \in V_2 \ \forall v \in V_1.$$

From our assumptions it follows that T and T_* are compact operators (Babuška and Osborn 1991); moreover, the adjoint of T on V_1 is given by $T^* = A^* \circ T_* \circ A^{*-1}$, where $A : V_1 \to V_2$ is the standard linear operator associated to the bilinear form a.

Remark 9.2. In some applications (for instance those involving spaces like $\mathbf{H}_0(\mathrm{div};\Omega)$ or $\mathbf{H}_0(\mathbf{curl};\Omega)$) it might be difficult to satisfy the compactness assumption on the bilinear form b. The theory can, however, be applied without modifications, by directly assuming the compactness of T and T_*.

A pair (λ, u) is an eigenmode of problem (9.2) if and only if it satisfies $\lambda T u = u$, that is, (μ, u) is an eigenpair of the operator T with $\mu = \lambda^{-1}$. The concepts of ascent multiplicity, algebraic multiplicity and generalized eigenfunctions of problem (9.2) are then defined in terms of the analogous properties for the operator T.

We shall also make use of the following adjoint eigenvalue problem: find $\lambda \in \mathbb{C}$ and $u \in V_2$, with $u \neq 0$, such that

$$a(v, u) = \lambda b(v, u) \quad \forall v \in V_1. \tag{9.3}$$

The discretization of problem (9.2) consists in selecting finite-dimensional subspaces $V_{1,h}$ and $V_{2,h}$, and in considering the following problem: find $\lambda_h \in \mathbb{C}$ and $v_{1,h} \in V_{1,h}$ with $v_{1,h} \neq 0$ such that

$$a(v_{1,h}, v_2) = \lambda_h b(v_{1,h}, v_2) \quad \forall v_2 \in V_{2,h}. \tag{9.4}$$

We suppose that

$$\dim(V_{1,h}) = \dim(V_{2,h}),$$

so that (9.4) is actually a generalized (square) eigenvalue problem.

We assume that the discrete uniform inf-sup conditions are satisfied,

$$\inf_{v_1 \in V_{1,h}} \sup_{v_2 \in V_{2,h}} \frac{|a(v_1, v_2)|}{\|v_1\|_{V_1} \|v_2\|_{V_2}} \geq \gamma > 0,$$
$$\sup_{v_1 \in V_{1,h}} |a(v_1, v_2)| > 0 \quad \forall v_2 \in V_{2,h} \setminus \{0\},$$

so that the discrete solution operators T_h and $T_{*,h}$ can be defined in analogy to T and T_*. It is clear that the convergence of the eigensolutions of (9.4) towards those of (9.2) can be analysed by means of the convergence of T_h and $T_{*,h}$ to T and T_*.

We are now ready to report the four main results of the theory. For each result, we state a theorem concerning the approximation of the eigenpairs of T followed by a corollary containing the consequences for the approximation of the eigensolutions to (9.2).

We consider an eigenvalue λ of (9.2) ($\mu = \lambda^{-1}$ in the case of the operator T) of algebraic multiplicity m and with ascent of $\mu - T$ equal to α. For the sake of generality, we let X denote the domain V_1 of the operator T, so we shall revert to the notation of Section 6.

The first theorem concerns the approximation of eigenvectors.

Theorem 9.3. Let μ be a non-zero eigenvalue of T, let $E = E(\mu)X$ be its generalized eigenspace, and let $E_h = E_h(\mu)X$. Then

$$\hat{\delta}(E, E_h) \leq C\|(T - T_h)_{|E}\|_{\mathcal{L}(X)}.$$

Corollary 9.4. Let λ be an eigenvalue of (9.2), let $E = E(\lambda^{-1})V_1$ be its generalized eigenspace and let $E_h = E_h(\lambda^{-1})V_1$. Then

$$\hat{\delta}(E, E_h) \leq C \sup_{\substack{u\in E\\ \|u\|=1}} \inf_{v\in V_{1,h}} \|u - v\|_{V_1}.$$

In the case of multiple eigenvalues it has been observed that it is convenient to introduce the arithmetic mean of the approximating eigenvalues.

Theorem 9.5. Let μ be a non-zero eigenvalue of T with algebraic multiplicity equal to m and let $\widehat{\mu}_h$ denote the arithmetic mean of the m discrete eigenvalues of T_h converging towards μ. Let $\phi_1, \dots, \phi_m$ be a basis of generalized eigenvectors in $E = E(\mu)X$ and let $\phi_1^*, \dots, \phi_m^*$ be a dual basis of generalized eigenvectors in $E^* = E^*(\overline{\mu})X$. Then

$$|\mu - \widehat{\mu}_h| \leq \frac{1}{m}\sum_{i=1}^{m} |((T - T_h)\phi_i, \phi_i^*)| + C\|(T - T_h)_{|E}\|_{\mathcal{L}(X)}\|(T^* - T_h^*)_{|E^*}\|_{\mathcal{L}(X)}.$$

Corollary 9.6. Let λ be an eigenvalue of (9.2) and let $\widehat{\lambda}_h$ denote the arithmetic mean of the m discrete eigenvalues of (9.4) converging towards λ. Then

$$|\lambda - \widehat{\lambda}_h| \leq C \sup_{\substack{u\in E\\ \|u\|=1}} \inf_{v\in V_{1,h}} \|u - v\|_{V_1} \sup_{\substack{u\in E^*\\ \|u\|=1}} \inf_{v\in V_{2,h}} \|u - v\|_{V_2},$$

where E is the space of generalized eigenfunctions associated with λ and E^* is the space of generalized adjoint eigenfunctions associated with λ (see the adjoint problem (9.3)).

The estimate of the error in the eigenvalues involves the ascent multiplicity α.

Theorem 9.7. Let $\phi_1, \dots, \phi_m$ be a basis of the generalized eigenspace $E = E(\mu)X$ of T and let $\phi_1^*, \dots, \phi_m^*$ be a dual basis. Then, for $i = 1, \dots m$,

$$|\mu - \mu_{i,h}|^\alpha \leq \\ C\left\{\sum_{j,k=1}^{m} |((T - T_h)\phi_j, \phi_k^*)| + \|(T - T_h)_{|E}\|_{\mathcal{L}(X)}\|(T^* - T_h^*)_{|E^*}\|_{\mathcal{L}(X)}\right\},$$

where $\mu_{1,h}, \dots, \mu_{m,h}$ are the m discrete eigenvalues (repeated according to their algebraic multiplicity) converging to μ, and E^* is the space of generalized eigenvectors of T^* associated with $\overline{\mu}$.

Corollary 9.8. With notation analogous to that of the previous theorem, for $i = 1, \ldots, m$ we have

$$|\lambda - \lambda_{i,h}|^\alpha \leq C \sup_{\substack{u \in E \\ \|u\|=1}} \inf_{v \in V_{1,h}} \|u - v\|_{V_1} \sup_{\substack{u \in E^* \\ \|u\|=1}} \inf_{v \in V_{2,h}} \|u - v\|_{V_2}, \tag{9.5}$$

where E is the space of generalized eigenfunctions associated with λ and E^* is the space of generalized adjoint eigenfunctions associated with λ (see the adjoint problem (9.3)).

Remark 9.9. Apparently, the estimates of Corollaries 9.6 and 9.8 are not immediate consequences of Theorems 9.5 and 9.7. In Section 10 we give a proof of these results in the particular case of the Laplace eigenvalue problem. The interested reader is referred to Babuška and Osborn (1991) for the general case.

The last result is more technical than the previous ones and complements Theorem 9.3 on the description of the approximation of the generalized eigenvectors. In particular, for $k = \ell = 1$, the theorem applies to the eigenvectors.

Theorem 9.10. Let $\{\mu_h\}$ be a sequence of discrete eigenvalues of T_h converging to a non-zero eigenvalue μ of T. Consider a sequence $\{u_h\}$ of unit vectors in $\ker(\mu_h - T_h)^k$ for some $k \leq \alpha$ (discrete generalized eigenvectors of order k). Then, for any integer ℓ with $k \leq \ell \leq \alpha$, there exists a generalized eigenvector $u(h)$ of T of order ℓ such that

$$\|u(h) - u_h\|_X^{\alpha/(\ell-k+1)} \leq C \|(T - T_h)_{|E}\|_{\mathcal{L}(X)}.$$

Corollary 9.11. Let $\{\lambda_h\}$ be a sequence of discrete eigenvalues of (9.4) converging to an eigenvalue λ of (9.2). Consider a sequence $\{u_h\}$ of unit eigenfunctions in $\ker(\lambda_h^{-1} - T_h)^k$ for some $k \leq \alpha$ (discrete generalized eigenfunctions of order k). Then, for any integer ℓ with $k \leq \ell \leq \alpha$, there exists a generalized eigenvector $u(h)$ of (9.2) of order ℓ such that

$$\|u(h) - u_h\|_{V_1}^{\alpha/(\ell-k+1)} \leq C \sup_{\substack{u \in E \\ \|u\|=1}} \inf_{v \in V_{1,h}} \|u - v\|_{V_1}.$$

We conclude this section with the application of the present theory to the case of symmetric variationally posed eigenvalue problems. In particular, the presented results will give a more comprehensive and precise treatment of the problems discussed in Sections 7 and 8.

We suppose that $V_1 = V_2$ are identical real Hilbert spaces, denoted by V, and that a and b are real and symmetric. We assume that a is V-elliptic,

$$a(v,v) \geq \gamma > 0 \quad \forall v \in V,$$

and that b can be extended to a continuous bilinear form on $H \times H$ with a Hilbert space H such that $V \subset H$ is a compact inclusion. We assume, moreover, that b is positive on $V \times V$:

$$b(v,v) > 0 \quad \forall v \in V \setminus \{0\}.$$

In this case, as has already been observed in Section 7, the eigenvalues of (9.2) are positive and can be ordered in a sequence tending to infinity,

$$0 < \lambda^{(1)} \leq \lambda^{(2)} \leq \cdots \leq \lambda^{(k)} \leq \cdots,$$

where we repeat the eigenvalues according to their multiplicities (we recall that geometric and algebraic multiplicities are now the same and that the ascent multiplicity of $1/\lambda^{(k)} - T$ is 1 for all k).

Let $V_h \subset V$ be the finite-dimensional space used for the eigenmodes approximation and denote by $\lambda_h^{(k)}$ ($k = 1, \ldots, \dim(V_h)$) the discrete eigenvalues. The min-max principle (see Section 7, in particular Proposition 7.2 and the discussion thereafter) and Corollary 9.8 give the following result.

Theorem 9.12. For each k, we have

$$\lambda^{(k)} \leq \lambda_h^{(k)} \leq \lambda^{(k)} + C \sup_{\substack{u \in E \\ \|u\|=1}} \inf_{v \in V_h} \|u - v\|_V^2,$$

where E denotes the eigenspace associated with $\lambda^{(k)}$.

Corollary 9.4 reads as follows in the symmetric case.

Theorem 9.13. Let $u^{(k)}$ be a unit eigenfunction associated with an eigenvalue $\lambda^{(k)}$ of multiplicity m, such that $\lambda^{(k)} = \cdots = \lambda^{(k+m-1)}$, and denote by $u_h^{(k)}, \ldots, u_h^{(k+m-1)}$ the eigenfunctions associated with the m discrete eigenvalues converging to $\lambda^{(k)}$. Then, there exists

$$w_h^{(k)} \in \operatorname{span}\{u_h^{(k)}, \ldots, u_h^{(k+m-1)}\}$$

such that

$$\|u^{(k)} - w_h^{(k)}\|_V \leq C \sup_{\substack{u \in E \\ \|u\|=1}} \inf_{v \in V_h} \|u - v\|_V,$$

where E denotes the eigenspace associated with $\lambda^{(k)}$.

The results of the present section contain the basic estimates for eigenvalues and eigenfunctions of compact variationally posed eigenvalue problems. Several other refined results are available.

For instance, it is possible to obtain sharper estimates in the case of multiple eigenvalues (see Knyazev and Osborn (2006) and the references therein): in particular, these can be useful when a multiple eigenvalue is associated with eigenfunctions with different regularities. Estimate (9.5) would predict that in such a case the eigenvalue is approximated with the order of convergence dictated by the lowest regularity of the eigenfunctions; on the other hand it is possible that the approximating eigenvalues have different speeds according to the different regularities of the associated eigenfunctions.

10. The Laplace eigenvalue problem

In this section we apply the Babuška–Osborn theory to the convergence analysis of conforming finite element approximation to Laplace eigenvalue problem.

The Laplace eigenvalue problem has already been analysed in several parts of this paper, but we recall here the related variational formulations for completeness. Given $\Omega \subset \mathbb{R}^n$ and the real Sobolev space $H^1_0(\Omega)$, we look for eigenvalues $\lambda \in \mathbb{R}$ and eigenfunctions $u \in H^1_0(\Omega)$, with $u \neq 0$, such that

$$(\mathbf{grad}\, u, \mathbf{grad}\, v) = \lambda(u, v) \quad \forall v \in H^1_0(\Omega).$$

The Riesz–Galerkin discretization makes use of a finite-dimensional space $V_h \subset H^1_0(\Omega)$, and consists in looking for eigenvalues $\lambda_h \in \mathbb{R}$ and eigenfunctions $u_h \in V_h$, with $u_h \neq 0$, such that

$$(\mathbf{grad}\, u_h, \mathbf{grad}\, v) = \lambda_h(u_h, v) \quad \forall v \in V_h.$$

We denote by $a(\cdot, \cdot)$ the bilinear form $(\mathbf{grad}\, \cdot, \mathbf{grad}\, \cdot)$ and remark that the considerations of this section can be easily generalized to any bilinear form a which is equivalent to the scalar product of $H^1_0(\Omega)$. Moreover, other types of homogeneous boundary conditions might be considered as well.

Using the notation of the previous section, the starting point for the analysis consists in a suitable definition of the solution operator $T : X \to X$ and, in particular, of the functional space X. Let V and H denote the spaces $H^1_0(\Omega)$ and $L^2(\Omega)$, respectively. The first, natural, definition consists in taking $X = V$ and in defining $T : V \to V$ by

$$a(Tf, v) = (f, v) \quad \forall v \in V. \tag{10.1}$$

Of course, the above definition can be easily extended to $X = H$, since it makes perfect sense to consider the solution to the source Laplace problem with f in $L^2(\Omega)$. We then have at least two admissible definitions: $T_V : V \to V$ and $T_H : H \to H$. Clearly, T_H can be defined analogously as for T_V by $T_H f \in V \subset H$ and (10.1): the only difference between T_V and T_H is the underlying spaces.

It is clear that T_V and T_H are self-adjoint. Since we are dealing with a basic example, we stress the details of this result.

Lemma 10.1. If V is endowed with the norm induced by the scalar product given by the bilinear form a, then the operator T_V is self-adjoint.

Proof. The result follows from the identities

$$a(T_V x, y) = (x, y) = (y, x) = a(T_V y, x) = a(x, T_V y) \quad \forall x, y \in V. \qquad \square$$

Lemma 10.2. The operator T_H is self-adjoint.

Proof. The result follows from the identities

$$(T_H x, y) = (y, T_H x) = a(T_H y, T_H x) = a(T_H x, T_H y) = (x, T_H y) \quad \forall x, y \in H.$$
$\square$

Moreover, it is clear that the eigenvalues/eigenfunctions of the operators T_V and T_H coincide, so that either operator can be used for the analysis.

The discussion of Section 9 shows that two main steps are involved. First of all we have to define a suitable discrete solution operator T_h satisfying a convergence in norm results like (9.1). As a second step, only after we know that the eigenvalues/eigenfunctions are well approximated can we estimate the order of convergence.

For the convergence analysis performed in the next sections, we assume that V_h is such that the following best approximation holds:

$$\inf_{v \in V_h} \|u - v\|_{L^2(\Omega)} \le C h^{\min\{k+1, r\}} \|u\|_{H^r(\Omega)},$$
$$\inf_{v \in V_h} \|u - v\|_{H^1(\Omega)} \le C h^{\min\{k, r-1\}} \|u\|_{H^r(\Omega)}.$$

Such estimates are standard when V_h contains piecewise polynomials of degree k.

10.1. Analysis for the choice $T = T_V$

We now show how to use the results of Section 9 with the choice $T = T_V$. The discrete solution operator can be defined in a coherent way as $T_h : V \to V$ by $T_h f \in V_h \subset V$, and

$$a(T_h f, v) = (f, v) \quad \forall v \in V_h.$$

The standard error estimate for the solution of the *source* Laplace equation implies that the norm convergence (9.1) is satisfied for all reasonable domains. If Ω is Lipschitz-continuous, for instance, the following estimate is well known: there exists $\varepsilon > 0$ such that

$$\|Tf - T_h f\|_{H^1_0(\Omega)} \le C h^{\varepsilon} \|f\|_{H^1_0(\Omega)}.$$

We can then conclude that the consequences of Theorem 9.1 are valid: all continuous eigenvalues/eigenfunctions are correctly approximated and all discrete eigenvalues/eigenfunctions approximate some continuous eigenvalues/eigenfunctions with the correct multiplicity.

We now come to the task of estimating the rate of convergence. Let us suppose that we are interested in the convergence rate for the approximation of the eigenvalue λ, and that the regularity of the eigenspace E associated with λ is r, that is, $E \subset H^r(\Omega)$, which implies

$$\|(T - T_h)_{|E}\|_{\mathcal{L}(V)} = O(h^{\min\{k,r-1\}}). \tag{10.2}$$

Let us denote by τ the quantity $\min\{k, r-1\}$.

The estimate for the eigenfunctions is a more-or-less immediate consequence of Corollary 9.4: we easily deduce the result of Theorem 9.13, which can be summarized in this case by the next theorem.

Theorem 10.3. Let u be a unit eigenfunction associated with the eigenvalue λ of multiplicity m, and let $w_h^{(1)}, \dots, w_h^{(m)}$ denote eigenfunctions associated with the m discrete eigenvalues converging to λ. Then there exists

$$u_h \in \operatorname{span}\{w_h^{(1)}, \dots, w_h^{(m)}\}$$

such that

$$\|u - u_h\|_V \le C h^\tau \|u\|_{H^{1+\tau}(\Omega)}.$$

We now see how the result of Theorem 9.12 can be obtained from Theorem 9.7. The proposed arguments will also provide a proof of Corollary 9.8 in this particular case.

Theorem 10.4. Let λ_h be an eigenvalue converging to λ. Then the following optimal double order of convergence holds:

$$\lambda \le \lambda_h \le \lambda + C h^{2\tau}.$$

Proof. From (10.2) and the conclusion of Theorem 9.7 it is clear that, since T is self-adjoint, we only need to bound the term

$$\sum_{j,k=1}^{m} |((T - T_h)\phi_j, \phi_k)_V|,$$

where $\{\phi_1, \dots, \phi_m\}$ is a basis for the eigenspace E.

We have

$$\begin{aligned} |((T - T_h)u, v)_V| &\le C|a((T - T_h)u, v)| = C \inf_{v_h \in V_h} |a((T - T_h)u, v - v_h)| \\ &\le \|(T - T_h)u\|_V \inf_{v_h \in V_h} \|v - v_h\|_V \\ &\le C h^\tau \|u\|_V h^\tau \|v\|_{H^{1+\tau}(\Omega)} \le C h^{2\tau} \|u\|_V \|v\|_V, \end{aligned}$$

which is valid for any $u, v \in E$ since $v = \lambda T v$ implies

$$\|v\|_{H^{1+\tau}(\Omega)} \le C \|v\|_V. \qquad \square$$

10.2. Analysis for the choice $T = T_H$

If we choose to perform our analysis in the space $H = L^2(\Omega)$, then we have to define the discrete operator $T_h : H \to H$, which can be done by taking $T_h f \in V_h \subset V \subset H$ as

$$a(T_h f, v) = (f, v) \quad \forall v \in V_h.$$

As in the previous case, it is immediate to obtain that the convergence in norm (9.1) is satisfied for any reasonable domain. Namely, for Ω Lipschitz-continuous, we have that there exists $\varepsilon > 0$ with

$$\|Tf - T_h f\|_{L^2(\Omega)} \le C h^{1+\varepsilon} \|f\|_{L^2(\Omega)}.$$

Using the same definition of τ as in the previous case, we can easily deduce from Theorem 9.13 the optimal convergence estimate for the eigenfunctions.

Theorem 10.5. Let u be a unit eigenfunction associated with the eigenvalue λ of multiplicity m and let $w_h^{(1)}, \dots, w_h^{(m)}$ denote eigenfunctions associated with the m discrete eigenvalues converging to λ. Then there exists $u_h \in \operatorname{span}\{w_h^{(1)}, \dots, w_h^{(m)}\}$ such that

$$\|u - u_h\|_H \le C h^{1+\tau} \|u\|_{H^{1+\tau}(\Omega)}.$$

We conclude this section by showing that the estimates already proved for the eigenvalues (optimal double order of convergence) and the eigenfunctions (optimal order of convergence in $H^1(\Omega)$) can also be obtained in this setting. In order to get a rate of convergence for the eigenvalues, we need, as in the proof of Theorem 10.4, an estimate for the term

$$\sum_{j,k=1}^{m} |((T - T_h)\phi_j, \phi_k)_H|,$$

where $\{\phi_1, \dots, \phi_m\}$ is a basis for the eigenspace E.

The conclusion is a consequence of the following estimate:

$$\begin{aligned}
|((T - T_h)u, v)| &= |(v, (T - T_h)u)| = |a(Tv, (T - T_h)u)| \\
&= |a((T - T_h)u, Tv)| = |a((T - T_h)u, Tv - T_h v)| \\
&\le \|(T - T_h)u\|_V \|(T - T_h)v\|_V \\
&\le C h^{2\tau},
\end{aligned}$$

which is valid for any $u, v \in E$ with $\|u\|_H = \|v\|_H = 1$.

Finally, using the same notation as in Theorem 10.5, the estimate in V for the eigenfunctions associated with *simple* eigenvalues follows from the identity

$$a(u - u_h, u - u_h) = \lambda(u - u_h, u - u_h) - (\lambda - \lambda_h)(u_h, u_h),$$

which can be obtained directly from the definitions of u, u_h, λ, and λ_h. The case of eigenfunctions with higher multiplicity is less immediate, but can be handled with similar tools, using for λ_h a suitable linear combination of the discrete eigenvalues converging to λ.

11. Non-conforming approximation of eigenvalue problems

The aim of this section is to see how the theory developed so far changes when it is applied to non-conforming approximations, in particular when V_h is not contained in V. Our interest in this topic lies in the fact that mixed discretizations of partial differential equations can be seen as particular situations of non-conforming approximations. We shall devote Part 3 of this paper to the analysis of mixed finite elements for eigenvalue problems.

The question of the non-conforming approximation of compact eigenvalue problems has been raised often in the literature, and several possible answers are available. Without attempting to be complete, we refer the interested reader to Rannacher (1979), Stummel (1980), Werner (1981), Armentano and Durán (2004) and Alonso and Dello Russo (2009). Non-conforming approximations can also be analysed in the nice setting introduced in Descloux, Nassif and Rappaz (1978*a*, 1978*b*) for the approximation of non-compact operators.

We start with a basic example: triangular Crouzeix–Raviart elements for the Laplace eigenvalue. This example has already been discussed from the numerical point of view in Table 3.5. For this example, probably the most complete reference can be found in Durán, Gastaldi and Padra (1999), where this element is studied for the solution of an auxiliary problem. The continuous problem is the same as in the previous section: find $\lambda \in \mathbb{R}$ and $u \in V$ with $u \neq 0$ such that

$$a(u, v) = \lambda(u, v) \quad \forall v \in V,$$

with $V = H^1_0(\Omega)$ and $a(u, v) = (\mathbf{grad}\, u, \mathbf{grad}\, v)$. More general situations might be considered with the same arguments; in particular, the analysis is not greatly affected by the presence of a generic elliptic bilinear form a or the case of a right-hand side of the equation where the the scalar product in $L^2(\Omega)$ is replaced by a bilinear form b which is equivalent to the scalar product (this apparently small change may, however, introduce an additional source of non-conformity). Other non-conforming finite elements might be considered as well, as long as suitable estimates for the consistency terms we are going to introduce are available.

Let V_h be the space of lowest-order Crouzeix–Raviart elements: given a triangular mesh $\mathcal{T}_h$ of the domain Ω with mesh size h, the space V_h consists of piecewise linear elements which are continuous at the midpoint of the inter-elements. The discrete problem is as follows: find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$

with $u_h \neq 0$ such that

$$a_h(u_h, v) = \lambda_h(u_h, v) \quad \forall v \in V_h,$$

where the *discrete* bilinear form a_h is defined as

$$a_h(u, v) = \sum_{K \in \mathcal{T}_h} \int_K \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}\mathbf{x} \quad \forall u, v \in V + V_h.$$

It is clear that $a_h(u, v) = a(u, v)$ if u and v belong to V. It is natural to introduce a discrete energy norm on the space $V + V_h$:

$$\|v\|_h^2 = \|v\|_{L^2(\Omega)}^2 + a_h(v, v) \quad \forall v \in V + V_h.$$

In the previous section we saw that the Babuška–Osborn theory can be applied to the analysis of the approximation of the Laplace eigenvalue problem with two different choices of the solution operator T. The first (and more standard) approach consists in choosing $T : V \to V$, while the second one makes use of $H = L^2(\Omega)$ and defines $T : H \to H$. It is clear that for the non-conforming approximation the first approach cannot produce any useful result, since it is impossible to construct an operator valued in V which represents the solution to our discrete problem which is defined in $V_h \not\subset V$. We shall then use the latter approach and define $T : H \to H$ as $Tf \in V$ given by

$$a(Tf, v) = (f, v) \quad \forall f \in H \text{ and } \forall v \in V.$$

The corresponding choice for the discrete operator $T_h : H \to H$ is $T_h f \in V_h \subset H$ given by

$$a_h(T_h f, v) = (f, v) \quad \forall f \in H \text{ and } \forall v \in V_h.$$

It is well known that T_h is well-defined, and that the following optimal estimate holds:

$$\|(T - T_h)f\|_h \leq Ch\|f\|_{L^2(\Omega)}. \tag{11.1}$$

The optimal estimate of $(T - T_h)f$ in $L^2(\Omega)$ requires more regularity than simply $f \in L^2(\Omega)$ (Durán *et al.* 1999, Lemmas 1 and 2). In general we have

$$\|(T - T_h)f\|_{L^2(\Omega)} \leq Ch^2\|f\|_{H^1(\Omega)},$$

which gives the optimal estimate if f is an eigenfunction, since in that case $\|f\|_{H^1(\Omega)}$ can be bounded by $\|f\|_{L^2(\Omega)}$:

$$\|(T - T_h)f\|_{L^2(\Omega)} \leq Ch^2\|f\|_{L^2(\Omega)} \quad \forall f \text{ eigenfunction of } T. \tag{11.2}$$

Remark 11.1. For the sake of simplicity, in the last estimates and in the following analysis we are assuming that the domain is convex, so we do not have to worry about the regularity of the solutions. Our arguments, however, cover the general case of lower regularity as well.

Estimate (11.1), in particular, shows that the convergence in norm (9.1) is satisfied with $X = H$, so we can conclude that the discrete eigenmodes converge to the continuous ones in the spirit of Theorem 9.1.

Let us now study the rate of convergence of the eigenvalues and eigenfunctions. Following what we have done in the previous section, we start with the analysis involving the eigenfunctions.

Theorem 11.2. Let u be a unit eigenfunction associated with the eigenvalue λ of multiplicity m and let $w_h^{(1)}, \dots, w_h^{(m)}$ denote linearly independent eigenfunctions associated with the m discrete eigenvalues converging to λ. Then there exists $u_h \in \operatorname{span}\{w_h^{(1)}, \dots, w_h^{(m)}\}$ such that

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^2 \|u\|_{L^2(\Omega)}.$$

Proof. The proof is an immediate consequence of Theorem 9.3 and estimate (11.2). □

Theorem 11.3. Let λ_h be an eigenvalue converging to λ. Then the following optimal double order of convergence holds:

$$|\lambda - \lambda_h| \leq Ch^2.$$

Proof. We are going to use Theorem 9.7 with $X = H$. It is clear that in our case T and T_h are self-adjoint, so we have $T^* = T$, $T_h^* = T_h$, and $\alpha = 1$. The second term in the estimate of Theorem 9.7 is of order h^4 due to (11.2), so we analyse in detail the term $((T - T_h)\phi_j, \phi_k)$. Let u and v be eigenfunctions associated with λ; we have to estimate $((T - T_h)u, v)$. It is clear that we can use the direct estimate

$$\|((T - T_h)u, v)\|_H \leq \|(T - T_h)u\|_H \|v\|_H,$$

and the result follows from (11.2).

We now present an alternative estimate of the term $((T - T_h)u, v)$ which emphasizes the role of the consistency error, and offers more flexibility for generalization to other types of non-conforming approximations,

$$\begin{aligned} ((T - T_h)u, v) &= a_h((T - T_h)u, Tv) + a_h(T_h u, (T - T_h)v) \\ &= a_h((T - T_h)u, (T - T_h)v) \\ &\quad + a_h((T - T_h)u, T_h v) + a_h(T_h u, (T - T_h)v). \end{aligned} \tag{11.3}$$

The first term on the right-hand side of (11.3) is of order h^2 from (11.1), so we need to bound the second term, which is analogous to the third one from the symmetry of a_h. We have

$$\begin{aligned} a_h((T - T_h)u, T_h v) &= a_h(Tu, T_h v) - (u, T_h v) \\ &= \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\mathbf{grad}\, Tu \cdot \mathbf{n}) T_h v, \end{aligned} \tag{11.4}$$

which by standard arguments is equal to

$$\sum_{e\in\mathcal{E}_h}\int_e \big((\mathbf{grad}\,Tu\cdot\mathbf{n}_e)-P_e(\mathbf{grad}\,Tu\cdot\mathbf{n}_e)\big)(T_hv-P_eT_hv)$$
$$=\sum_{e\in\mathcal{E}_h}\int_e \big((\mathbf{grad}\,Tu\cdot\mathbf{n}_e)-P_e(\mathbf{grad}\,Tu\cdot\mathbf{n}_e)\big)$$
$$\times\big((T_hv-P_eT_hv)-(Tv-P_eTv)\big),$$

where the set $\mathcal{E}_h$ contains all edges of the triangulation $\mathcal{T}_h$: the internal edges are repeated twice with opposite orientation of the normal and with appropriate definition of $T_hv_{|e}$ (which jumps from one triangle to the other); P_e denotes the $L^2(e)$-projection onto constant functions on e. We deduce

$$|a_h((T-T_h)u,T_hv)| \le \sum_{e\in\mathcal{E}_h}\|(I-P_e)(\mathbf{grad}\,Tu\cdot\mathbf{n}_e)\|_{L^2(e)}\|(I-P_e)(Tv-T_hv)\|_{L^2(e)}.$$

Putting all the pieces together gives the desired results from

$$\sum_{e\in\mathcal{E}_h}\|(I-P_e)(\mathbf{grad}\,Tu\cdot\mathbf{n}_e)\|_{L^2(e)}\le Ch^{1/2}\sum_{K\in\mathcal{T}_h}\|Tu\|_{H^2(K)}$$

and

$$\sum_{e\in\mathcal{E}_h}\|(I-P_e)(Tv-T_hv)\|_{L^2(e)}\le Ch^{1/2}\|(T-T_h)v\|_h \le Ch^{3/2}\|v\|_{L^2(\Omega)}.$$ □

We now deduce an optimal error estimate for the eigenfunctions in the discrete energy norm.

Theorem 11.4. With the same notation as in Theorem 11.2, we have

$$\|u-u_h\|_h\le Ch\|u\|_{L^2(\Omega)}.$$

Proof. We consider the case of a simple eigenvalue λ. The generalization to multiple eigenvalues is technical and needs no significant new arguments. We have the identity

$$u-u_h=\lambda Tu-\lambda_hT_hu_h = (\lambda-\lambda_h)Tu+\lambda_h(T-T_h)u+\lambda_hT_h(u-u_h),$$

and hence

$$\|u-u_h\|_h\le|\lambda-\lambda_h|\|Tu\|_{H^1(\Omega)}+\lambda_h\|(T-T_h)u\|_h+\lambda_h\|T_h(u-u_h)\|_h.$$

The first two terms are easily bounded by Theorem 11.3 and (11.1), respectively. The last term can be estimated by observing that we have

$$C\|T_h(u-u_h)\|_h^2\le a_h(T_h(u-u_h),T_h(u-u_h))=(u-u_h,T_h(u-u_h))$$

and using the estimate for $\|u-u_h\|_{L^2(\Omega)}$ in Theorem 11.2. □

PART THREE
Approximation of eigenvalue problems in mixed form

In this part we study the approximation of eigenvalue problems which have a particular structure, and which are often referred to as eigenvalue problems in mixed form.

12. Preliminaries

Given two Hilbert spaces Φ and Ξ, and two bilinear forms $a : \Phi \times \Phi \to \mathbb{R}$ and $b : \Phi \times \Xi \to \mathbb{R}$, the standard form of a *source* mixed problem is as follows: given $f \in \Phi'$ and $g \in \Xi'$, find $\psi \in \Phi$ and $\chi \in \Xi$ such that

$$a(\psi, \varphi) + b(\varphi, \chi) = \langle f, \varphi \rangle \quad \forall \varphi \in \Phi, \tag{12.1a}$$

$$b(\psi, \xi) = \langle g, \xi \rangle \quad \forall \xi \in \Xi. \tag{12.1b}$$

It is widely known that the natural conditions for the well-posedness of problem (12.1) are suitable inf-sup conditions, and that the discrete versions of the inf-sup conditions guarantee the stability of its approximation (Brezzi 1974, Babuška 1973, Brezzi and Fortin 1991).

If we suppose that there exist Hilbert spaces H_Φ and H_Ξ such that the following dense and continuous embeddings hold in a compatible way,

$$\Phi \subset H_\Phi \simeq H'_\Phi \subset \Phi',$$
$$\Xi \subset H_\Xi \simeq H'_\Xi \subset \Xi'$$

and we assume that the solution operator $T \in \mathcal{L}(H_\Phi \times H_\Xi)$ defined by

$$T(f, g) = (\psi, \chi) \tag{12.2}$$

(see (12.1)) is a compact operator, then a straightforward application of the theory presented in Part 2 might provide a straightforward convergence analysis for the approximation of the following eigenvalue problem: find $\lambda \in \mathbb{R}$ and $(\phi, \xi) \in \Phi \times \Xi$ with $(\phi, \xi) \neq (0, 0)$ such that

$$a(\psi, \varphi) + b(\varphi, \chi) = \lambda(\psi, \varphi)_{H_\Phi} \quad \forall \varphi \in \Phi,$$
$$b(\psi, \xi) = \lambda(\chi, \xi)_{H_\Xi} \quad \forall \xi \in \Xi.$$

On the other hand, the most common eigenvalue problems in mixed form correspond to a mixed formulation (12.1) where either f or g vanishes. Moreover, it is not obvious (the Laplace eigenvalue problem being the most famous counter-example) that the operator T defined on $H_\Phi \times H_\Xi$ is compact, so that, in general, eigenvalue problems in mixed form have to be studied with particular care.

Following Boffi, Brezzi and Gastaldi (1997), we consider two types of mixed eigenvalue problems. The first one (also known as the $(f, 0)$-type) consists of eigenvalue problems associated with the system (12.1) when $g = 0$. A fundamental example for this class is the Stokes eigenvalue problem. The second family (also known as the $(0, g)$-type) corresponds to the situation when in (12.1a) the right-hand side f vanishes. The Laplace eigenvalue problem in mixed form, for instance, belongs to this class.

The approximation of eigenvalue problems in mixed form has been the object of several papers; among them we refer to Canuto (1978), to Mercier, Osborn, Rappaz and Raviart (1981), which provides useful results for the estimate of the order of convergence, and to Babuška and Osborn (1991). The approach of this survey is taken from Boffi *et al.* (1997), where a comprehensive theory has been developed under the influence of the application presented in Boffi *et al.* (1999*b*) and the counter-example of Boffi *et al.* (2000*a*).

Since the notation used so far might appear cumbersome, in the next two sections we use notation which should resemble the Stokes problem in the case of problems of the first type and the mixed Laplace problem in the case of problems of the second type.

13. Problems of the first type

Let V, Q, and H be Hilbert spaces, suppose that the standard inclusions

$$V \subset H \simeq H' \subset V'$$

hold with continuous and dense embeddings, and let us consider two bilinear forms, $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$. We are interested in the following symmetric eigenvalue problem: find $\lambda \in \mathbb{R}$ and $u \in V$ with $u \neq 0$ such that, for some $p \in Q$,

$$a(u, v) + b(v, p) = \lambda(u, v) \quad \forall v \in V, \tag{13.1a}$$

$$b(u, q) = 0 \quad \forall q \in Q, \tag{13.1b}$$

where $(\cdot, \cdot)$ denotes the scalar product of H. We assume that a and b are continuous and that a is symmetric and positive semidefinite.

Given finite-dimensional subspaces $V_h \subset V$ and $Q_h \subset Q$, the discretization of (13.1) reads as follows: find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$ such that, for some $p_h \in Q$,

$$a(u_h, v) + b(v, p_h) = \lambda_h(u_h, v) \quad \forall v \in V_h, \tag{13.2a}$$

$$b(u_h, q) = 0 \quad \forall q \in Q_h. \tag{13.2b}$$

We start by studying the convergence of the eigensolution to (13.2) towards those of (13.1) and we shall discuss the rate of approximation afterwards. We aim to apply the spectral theory recalled in Section 6: in particular, we need to define a suitable solution operator. Let us consider the source problem associated with (13.1) (which corresponds to problem (12.1) with $g = 0$). Given $f \in H$, find $u \in V$ and $p \in Q$ such that

$$a(u, v) + b(v, p) = (f, v) \quad \forall v \in V, \tag{13.3a}$$
$$b(u, q) = 0 \qquad \forall q \in Q. \tag{13.3b}$$

Under the assumption that (13.3) is solvable for any $f \in H$ and that the component u of the solution is unique, we define $T : H \to V$ as $Tf = u$. We assume that

$$T \text{ is compact from } H \text{ to } V.$$

The discrete counterpart of (13.3) is as follows: find $u_h \in V_h$ and $p_h \in Q_h$ such that

$$a(u_h, v) + b(v, p_h) = (f, v) \quad \forall v \in V_h, \tag{13.4a}$$
$$b(u_h, q) = 0 \qquad \forall q \in Q_h. \tag{13.4b}$$

Assuming that the component u_h of the discrete solution of (13.4) exists and is unique, we can define the discrete operator $T_h : H \to V$ as $Tf = u_h \in V_h \subset V$.

It is clear that the eigenvalue problems (13.1) and (13.2), respectively, can be written in the equivalent form

$$\lambda T u = u, \quad \lambda_h T_h u_h = u_h.$$

We now introduce some abstract conditions that will guarantee the convergence of T_h to T in $\mathcal{L}(H, V)$. It follows from the discussion of Section 6 that this is a sufficient condition for the eigenmodes convergence. We denote by V_0 and Q_0 the subspaces of V and Q, respectively, containing all the solutions $u \in V$ and $p \in Q$ of (13.3) when f varies in H. In particular, we have $V_0 = T(H)$, and the inclusion $V_0 \subset \mathbb{K}$ holds true, where the kernel $\mathbb{K}$ of the operator associated with the bilinear form b is defined as usual by

$$\mathbb{K} = \{v \in V : b(v, q) = 0 \ \forall q \in Q\}.$$

We need also to introduce the discrete kernel as

$$\mathbb{K}_h = \{v_h \in V_h : b(v_h, q_h) = 0 \ \forall q_h \in Q_h\}.$$

It is well known that in general $\mathbb{K}_h \not\subset \mathbb{K}$. We shall also make use of suitable norms in V_0 and Q_0, which can be defined in a canonical way as

$$\|v\|_{V_0} = \inf\{\|f\|_H : Tf = v\},$$
$$\|q\|_{Q_0} = \inf\{\|f\|_H : q \text{ is the second component of the solution of (13.3)}\}.$$

Definition 13.1. The *ellipticity in the kernel* of the bilinear form a is satisfied if there exists $\alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|_V^2 \quad \forall v \in \mathbb{K}_h.$$

It can be shown that the ellipticity in the kernel property is sufficient for the well-posedness of the operator T_h (Boffi *et al.* 1997, Proposition 2).

Definition 13.2. We say that the *weak approximability* of Q_0 is satisfied if there exists $\rho_W(h)$, tending to zero as h tends to zero, such that

$$\sup_{v_h\in\mathbb{K}_h}\frac{b(v_h,q)}{\|v_h\|_V}\le\rho_W(h)\|q\|_{Q_0}\quad\forall q\in Q_0.$$

Definition 13.3. We say that the *strong approximability* of V_0 is satisfied if there exists $\rho_S(h)$, tending to zero as h tends to zero, such that, for any $v\in V_0$, there exists $v^I\in\mathbb{K}_h$ with

$$\|v-v^I\|_V\le\rho_S(h)\|v\|_{V_0}.$$

The next theorem says that the conditions introduced with the above definitions are sufficient for the eigenmode convergence of eigenvalue problems of the first kind.

Theorem 13.4. If the ellipticity in the kernel of the bilinear form a, the weak approximability of Q_0, and the strong approximability of V_0 are satisfied (see Definitions 13.1, 13.2 and 13.3), then there exists $\rho(h)$, tending to zero as h tends to zero, such that

$$\|(T-T_h)f\|_V\le\rho(h)\|f\|_H\quad\forall f\in H. \tag{13.5}$$

Proof. Take $f\in H$ and consider the solutions $(u,p)\in V_0\times Q_0$ of (13.3) and $(u_h,p_h)\in\mathbb{K}_h\times Q_h$ of (13.4) (p and p_h might not be unique). We need to estimate the difference $\|(T-T_h)f\|_V=\|u-u_h\|_V$. From the strong approximability of V_0, this can be performed by bounding the difference $\|u^I-u_h\|_V$. From the ellipticity in the kernel of a and the error equations, we have

$$\begin{aligned}\alpha\|u^I-u_h\|_V^2&\le a(u^I-u_h,u^I-u_h)\\&=a(u^I-u,u^I-u_h)+a(u-u_h,u^I-u_h)\\&=a(u^I-u,u^I-u_h)-b(u^I-u_h,p-p_h)\\&\le C\|u^I-u\|_V\|u^I-u_h\|_V-|b(u^I-u_h,p-p_h)|\\&\le\left(C\|u^I-u\|_V+\sup_{v_h\in\mathbb{K}_h}\frac{b(v_h,p-p_h)}{\|v_h\|_V}\right)\|u^I-u_h\|_V\\&\le\left(C\|u^I-u\|_V+\sup_{v_h\in\mathbb{K}_h}\frac{b(v_h,p)}{\|v_h\|_V}\right)\|u^I-u_h\|_V,\end{aligned}$$

which gives the required estimate thanks to the strong approximability of V_0 and the weak approximability of Q_0. □

Remark 13.5. The result of Theorem 13.4 can be essentially inverted by showing that the assumptions are necessary for the convergence in norm (13.5). This analysis is performed in Theorem 2 of Boffi *et al.* (1997), under the additional assumption that the operator T can be extended to a bounded operator in $\mathcal{L}(V', V)$.

Remark 13.6. So far, we have not assumed the well-posedness of the source problems (13.3) and (13.4). Indeed, we assumed existence and uniqueness of the first component of the solutions (u and u_h, respectively) in order to be able to define the solution operators T and T_h. On the other hand, the second component of solutions p and p_h might be non-unique. Examples of this situations (where p is unique, but p_h is not) are presented at the end of this section.

Remark 13.7. The presented result might look too strong, since we are considering the convergence in norm from H to V. Indeed, although our result immediately implies the convergence in norm from H into itself, it might be interesting to investigate directly the behaviour of $\|T - T_h\|$ in $\mathcal{L}(H)$ or $\mathcal{L}(V)$. On the other hand, the analysis presented in Theorem 13.4 is quite natural, and we are not aware of applications where a sharper result is needed.

Theorem 13.4 concerns good approximation of eigenvalues and eigenfunctions, but does not answer the question of estimating the rate of convergence. In most practical situations, this issue can be solved with the help of Babuška–Osborn theory, as developed in Section 9. This task was performed in Mercier *et al.* (1981) in the general situation of *non-symmetric* eigenvalue problems in mixed form; we report here the main results of this theory in the symmetric case (see Mercier *et al.* (1981, Section 5)).

Let λ be an eigenvalue of (13.1) of multiplicity m and let $E \subset V$ be the corresponding eigenspace. We denote by $\lambda_{1,h}, \lambda_{2,h}, \dots, \lambda_{m,h}$ the discrete eigenvalues converging to λ and by E_h the direct sum of the corresponding eigenspaces.

The convergence of eigenfunctions is a direct application of the results of the abstract theory and is summarized in the next theorem.

Theorem 13.8. Under the hypotheses of Theorem 13.4, there exists constant C such that

$$\hat{\delta}(E, E_h) \leq C\|(T - T_h)_{|E}\|_{\mathcal{L}(H,H)},$$

where the gap is evaluated in the H-norm.

Remark 13.9. The analogous estimate for the error in the norm of V can be obtained directly from Theorem 9.3 by assuming a uniform convergence of T_h to T in $\mathcal{L}(V)$. Such a convergence is usually a consequence of standard estimates for the source problem.

In order to estimate the convergence order for the eigenvalues, we consider the case when problems (13.3) and (13.4) are well-posed (see Remark 13.6). This is typically true if the standard inf-sup conditions are satisfied (Brezzi and Fortin 1991). According to Mercier *et al.* (1981), we can define the operator $B : H \to Q$ as $Bf = p$, where p is the second component of the solution of (13.3). Analogously, we define the discrete operator $B_h : H \to Q_h$ by $B_h f = p_h$, with p_h coming from (13.4). The results of Mercier *et al.* (1981, Theorem 5.1) are rewritten in the next theorem in this particular case.

Theorem 13.10. Under the hypotheses of Theorem 13.4 and the additional assumption that the operators B and B_h are well-defined, there exists C such that, for sufficiently small h,

$$\begin{aligned} |\lambda - \lambda_{i,h}| \le C\big(&\|(T-T_h)_{|E}\|^2_{\mathcal{L}(H,V)} \\ &+ \|(T-T_h)_{|E}\|_{\mathcal{L}(H,V)}\|(B-B_h)_{|E}\|_{\mathcal{L}(H,Q)}\big), \end{aligned} \tag{13.6}$$

for $i = 1, \dots, m$.

We conclude this section with some applications of the developed theory.

13.1. The Stokes problem

Let Ω be an open bounded domain in $\mathbb{R}^n$. The eigenvalue problem associated with the Stokes equations fits within the developed theory with the following definitions:

$$\begin{aligned} H &= L^2(\Omega), \\ V &= (H^1_0(\Omega))^n, \\ Q &= L^2_0(\Omega), \\ a(\mathbf{u},\mathbf{v}) &= \int_\Omega \underline{\varepsilon}(\mathbf{u}) : \underline{\varepsilon}(\mathbf{v})\,\mathrm{d}\mathbf{x}, \\ b(\mathbf{v},q) &= \int_\Omega q \operatorname{div}\mathbf{v}\,\mathrm{d}\mathbf{x}. \end{aligned}$$

It is well known that the bilinear form a is coercive in V, so the ellipticity in the kernel property (see Definition 13.1) holds true for any finite element choice. The weak approximability of Q_0 (see Definition 13.2) is satisfied by any reasonable approximating scheme as well. Namely, it can be easily seen that there exists $\varepsilon > 0$ such that the solution space Q_0 is contained in $H^\varepsilon(\Omega)$ for a wide class of domains (in particular, ε can be taken equal to one if Ω is a convex polygon or polyhedron). The weak approximability property then follows from standard approximation properties as follows:

$$|b(\mathbf{v}_h, q)| = |b(\mathbf{v}_h, q - q^I)| \le C\|\mathbf{v}_h\|_{H^1_0(\Omega)}\|q - q^I\|_{L^2(\Omega)},$$

where $q^I \in Q_h$ is any discrete function (recall that $\mathbf{v}_h \in \mathbb{K}_h$).

In this case the strong approximability of V_0 is the crucial condition for the convergence. In general V_0 consists of divergence-free functions which are in $(H^{1+\varepsilon}(\Omega))^n \cap V$ if the domain is smooth enough (for a suitable $\varepsilon > 0$, in particular $\varepsilon = 1$ if Ω is a convex polygon or polyhedron). The strong approximability means that, for any $\mathbf{v} \in V_0$, there exists $\mathbf{v}^I \in \mathbb{K}_h$ such that

$$\|\mathbf{v} - \mathbf{v}^I\|_{H_0^1(\Omega)} \leq \rho(h)\|\mathbf{v}\|_{V_0},$$

with $\rho(h)$ tending to zero as h tends to zero. This property is well known to be valid if, for instance, the discrete space V_h discretizes V with standard approximation properties and the classical inf-sup condition links V_h and Q_h: there exists $\beta > 0$ such that

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in V_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{H_0^1(\Omega)}\|q_h\|_{L^2(\Omega)}} > \beta.$$

This implies that all inf-sup stable approximations of the Stokes problem provide convergent discretization of the corresponding eigenvalue problem. As for the estimation of the rate of convergence (which was not considered in Boffi *et al.* (1997)), we can use Theorems 13.8 and 13.10. For instance, using a popular stable scheme of order k such as the generalized Hood–Taylor method (Boffi 1997), that is, continuous piecewise polynomials of degree k and $k-1$ for velocities and pressures, respectively, we get that the order of convergence of the eigenvalues is h^{2k}, as expected, if the domain is smooth enough in both two and three space dimensions. Indeed, looking at the terms appearing in formula (13.6), we essentially have to estimate two items: $\|(T - T_h)_{|E}\|_{\mathcal{L}(H,V)}$ and $\|(B - B_h)_{|E}\|_{\mathcal{L}(H,Q)}$. The classical error analysis for the source problem gives

$$\|\mathbf{u} - \mathbf{u}_h\|_V + \|p - p_h\|_Q \leq Ch^k\|f\|_H$$

if we assume that we have no limitation in the regularity of the solution $(\mathbf{u}, p)$. This implies

$$\|(T - T_h)_{|E}\|_{\mathcal{L}(H,V)} = O(h^k),$$
$$\|(B - B_h)_{|E}\|_{\mathcal{L}(H,Q)} = O(h^k),$$

which gives the result.

For the eigenfunctions, the result is similar. In particular, Theorem 13.8 and Remark 13.9 give that the gap between discrete and continuous eigenfunctions is of optimal order h^{k+1} in $L^2(\Omega)$ and h^k in $H^1(\Omega)$.

On the other hand, Theorem 13.4 allows us to conclude that we have the convergence of the eigenvalues even in cases when the source problem might not be solvable. This is the case, for instance, of the widely studied two-dimensional $\mathcal{Q}_1$–$\mathcal{P}_0$ element: that is, the velocities are continuous piecewise bilinear functions and the pressures are piecewise constants. It is well known that this element does not satisfy the inf-sup condition: a checkerboard spurious mode is present and the filtered inf-sup constant tends to zero as $O(h)$ (Johnson and Pitkäranta 1982). On the other hand, it can be proved that the hypotheses of Theorem 13.4 are satisfied (Boffi *et al.* 1997), so the eigenvalues are well approximated in spite of the fact that the approximation of the source problem is affected by spurious pressure modes.

13.2. Dirichlet problem with Lagrange multipliers

Following Babuška (1973), the Dirichlet problem with Lagrange multipliers can be studied with the theory developed in this section and the following identifications. Let Ω be a two-dimensional polygonal domain and define

$$\begin{aligned} H &= L^2(\Omega), \\ V &= H^1(\Omega), \\ Q &= H^{-1/2}(\partial\Omega), \\ a(u,v) &= \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}\mathbf{x}, \\ b(v,\mu) &= \langle \mu, v_{|\partial\Omega} \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$.

Given a regular decomposition of Ω and a regular decomposition of $\partial\Omega$, we let V_h be the space of continuous piecewise polynomials of degree k_1 defined in Ω, and let Q_h be the space of continuous piecewise polynomials of degree k_2 defined in $\partial\Omega$, with $k_1 \geq 0$ and $k_2 \geq 0$. It turns out that the weak approximability condition (see Definition 13.2) and the strong approximability condition (see Definition 13.3) are satisfied for any choice of triangulation and degrees k_1 and k_2. On the other hand, the ellipticity in the kernel property (see Definition 13.1) is satisfied under the weak assumption that Q_h contains μ_h such that $\langle \mu_h, 1 \rangle \neq 0$.

It is interesting to notice that in this example we have good convergence of the discretized eigenvalue problem under very general assumptions, while the corresponding source problem requires rather strict compatibility conditions between the meshes of Ω and $\partial\Omega$, and k_1 and k_2.

14. Problems of the second type

Let Σ, U, and H be Hilbert spaces, suppose that the standard inclusions

$$U \subset H \simeq H' \subset U'$$

hold with continuous and dense embeddings, and let us consider two bilinear forms $a : \Sigma \times \Sigma \to \mathbb{R}$ and $b : \Sigma \times U \to \mathbb{R}$. We are interested in the following symmetric eigenvalue problem: find $\lambda \in \mathbb{R}$ and $u \in U$ with $u \neq 0$ such that, for some $\sigma \in \Sigma$,

$$a(\sigma,\tau) + b(\tau,u) = 0 \quad \forall\tau \in \Sigma, \tag{14.1a}$$

$$b(\sigma,v) = -\lambda(u,v) \quad \forall v \in U, \tag{14.1b}$$

where $(\cdot,\cdot)$ denotes the scalar product in H. We assume that a and b are continuous and that a is symmetric and positive semidefinite. The hypotheses on a imply that the seminorm

$$|v|_a = (a(v,v))^{1/2}$$

is well-defined, so that we have

$$a(u,v) \leq |u|_a|v|_a \quad \forall u,v \in U.$$

Moreover, we assume that the following source problem associated with (14.1) has a unique solution $(\sigma,u) \in \Sigma \times U$ to

$$a(\sigma,\tau) + b(\tau,u) = 0 \quad \forall\tau \in \Sigma, \tag{14.2a}$$

$$b(\sigma,v) = -\langle g,v\rangle \quad \forall v \in U \tag{14.2b}$$

satisfying the *a priori* bound

$$\|\sigma\|_\Sigma + \|u\|_U \leq C\|g\|_{U'},$$

where the symbol $\langle\cdot,\cdot\rangle$ in (14.2) denotes the duality pairing between U' and U.

Given finite-dimensional subspaces $\Sigma_h \subset \Sigma$ and $U_h \subset U$, the Galerkin discretization of (14.1) reads as follows: find $\lambda_h \in \mathbb{R}$ and $u_h \in U_h$ with $u_h \neq 0$ such that, for some $\sigma_h \in \Sigma_h$,

$$a(\sigma_h,\tau) + b(\tau,u_h) = 0 \quad \forall\tau \in \Sigma_h, \tag{14.3a}$$

$$b(\sigma_h,v) = -\lambda_h(u_h,v) \quad \forall v \in U_h. \tag{14.3b}$$

Following the same lines as the previous section, we start by analysing how the solutions of (14.3) converge towards those of (14.1), and postpone the question of the rate of convergence to the end of this section.

We define the solution operator $T : H \to H$ by $Tg = u \in U \subset H$, where $u \in U$ is the second component of the solution to (14.2). It is clear that when g belongs to H the duality pairing in the right-hand side of (14.2) is equivalent to the scalar product (g,v).

The discrete counterpart of (14.2) when g belongs to H is as follows: find $(\sigma_h,u_h) \in \Sigma_h \times U_h$ such that

$$a(\sigma_h,\tau) + b(\tau,u_h) = 0 \quad \forall\tau \in \Sigma_h, \tag{14.4a}$$

$$b(\sigma_h,v) = -(g,v) \quad \forall v \in U_h. \tag{14.4b}$$

We suppose that the second component u_h of the solution of (14.4) exists and is unique, so we can define the discrete solution operator $T_h : H \to H$ as $T_hg = u_h \in U_h \subset U \subset H$.

We now introduce some abstract conditions that will be used in an initial theorem in order to show the convergence of T_h to T in $\mathcal{L}(H,H)$ and, in a second theorem, in $\mathcal{L}(H,U)$. According to the results of Section 9, this is enough to show the convergence of the eigensolutions of (14.3) towards those of (14.1). We let Σ_0 and U_0 denote the subspaces of Σ and U, respectively, containing all solutions $\sigma \in \Sigma$ and $u \in U$ of (14.2) when g varies in H. In particular, we have $U_0 = T(H)$. We shall also make use of the space $\Sigma_{0'} \subset \Sigma$ containing the second components of the solution $\sigma \in \Sigma$ of (14.2) when g varies in U'. The spaces Σ_0, U_0, and $\Sigma_{0'}$ will be endowed with their natural norms:

$$\begin{aligned}\|\tau\|_{\Sigma_0} &= \inf\{\|g\|_H : \tau \text{ is solution of (14.2) with datum } g\},\\ \|v\|_{U_0} &= \inf\{\|g\|_H : Tg = v\},\\ \|\tau\|_{\Sigma_{0'}} &= \inf\{\|g\|_{U'} : \tau \text{ is solution of (14.2) with datum } g\}.\end{aligned}$$

Finally, the discrete kernel is given by

$$\mathbb{K}_h = \{\tau_h \in \Sigma_h : b(\tau_h, v) = 0 \ \forall v \in U_h\}.$$

Definition 14.1. We say that the *weak approximability* of U_0 with respect to a is satisfied if there exists $\rho_W(h)$, tending to zero as h tends to zero, such that

$$b(\tau_h, v) \le \rho_W(h)|\tau_h|_a\|v\|_{U_0} \quad \forall v \in U_0 \ \forall \tau_h \in \mathbb{K}_h.$$

Definition 14.2. We say that the *strong approximability* of U_0 is satisfied if there exists $\rho_S(h)$, tending to zero as h tends to zero, such that, for every $v \in U_0$, there exists $v^I \in U_h$ such that

$$\|v - v^I\|_U \le \rho_S(h)\|v\|_{U_0}.$$

Remark 14.3. The same terms, *weak* and *strong approximability*, were used in the framework of problems of the first kind (see Definitions 13.2 and 13.3) and of the second kind (see Definitions 14.1 and 14.2). In the applications, it should be clear from the context which definition the terms refer to.

A powerful and commonly used tool for the analysis of mixed approximation is the Fortin operator (Fortin 1977), that is, a linear operator $\Pi_h : \Sigma_0 \to \Sigma_h$ that satisfies

$$b(\tau - \Pi_h\tau, v) = 0 \quad \forall \tau \in \Sigma_0 \ \forall v \in U_h. \tag{14.5}$$

Definition 14.4. A *bounded Fortin operator* is a Fortin operator that can be extended from Σ_0 to $\Sigma_{0'}$ and that is uniformly bounded in $\mathcal{L}(\Sigma_{0'}, \Sigma)$.

Definition 14.5. The *Fortid condition* is satisfied if there exists a Fortin operator which converges to the identity in the following sense. There exists $\rho_F(h)$, tending to zero as h tends to zero, such that

$$|\sigma - \Pi_h\sigma|_a \le \rho_F(h)\|\sigma\|_{\Sigma_0} \quad \forall\sigma \in \Sigma_0.$$

The next theorem presents sufficient conditions for good approximation of the eigensolutions of (14.3) towards those of (14.1).

Theorem 14.6. If the Fortid condition, the weak approximability of U_0 with respect to a, and the strong approximability of U_0 are satisfied (see Definitions 14.5, 14.1 and 14.2), then there exists $\rho(h)$, tending to zero as h tends to zero, such that

$$\|(T - T_h)g\|_H \le \rho(h)\|g\|_H \quad \forall g \in H.$$

Proof. Let g be in H and consider the solution (σ, u) of (14.2). In particular, we have $u = Tg$. Let us define $u_h = T_h g$ and let $\sigma_h \in \Sigma_h$ be such that (σ_h, u_h) solves (14.4) (such a σ_h might not be unique). Using a duality argument, let $(\sigma(h), u(h)) \in \Sigma \times U$ be the solution of (14.2) with $g = u - u_h \in U \subset H$. By the definition of the norm in Σ_0, we have $\|\sigma(h)\|_{\Sigma_0} \le \|u - u_h\|_H$. Moreover,

$$\begin{aligned}
\|u - u_h\|_H^2 &= (u - u_h, u - u_h) = -b(\sigma(h), u - u_h)\\
&= b(\sigma(h) - \Pi_h\sigma(h), u) + b(\Pi_h\sigma(h), u - u_h)\\
&= a(\sigma, \sigma(h) - \Pi_h\sigma(h)) + a(\sigma - \sigma_h, \Pi_h\sigma(h))\\
&\le |\sigma|_a|\sigma(h) - \Pi_h\sigma(h)|_a + |\sigma - \sigma_h|_a|\Pi_h\sigma(h)|_a\\
&\le |\sigma|_a|\sigma(h) - \Pi_h\sigma(h)|_a + C|\sigma - \sigma_h|_a\|\sigma(h)\|_{\Sigma_0}\\
&\le |\sigma|_a\rho_F(h)\|\sigma(h)\|_{\Sigma_0} + C|\sigma - \sigma_h|_a\|\sigma(h)\|_{\Sigma_0}\\
&\le (\rho_F(h)|\sigma|_a + C|\sigma - \sigma_h|_a)\|u - u_h\|_H,
\end{aligned}$$

where $\rho_F(h)$ was introduced in Definition 14.5. This implies

$$\|u - u_h\|_H \le \rho_F(h)|\sigma|_a + C|\sigma - \sigma_h|_a.$$

It remains to estimate the term $|\sigma - \sigma_h|_a$, which we do by triangular inequality after summing and subtracting $\Pi_h\sigma$. Since we assumed the Fortid condition, it is enough to bound $|\Pi_h\sigma - \sigma_h|_a$. We notice that $|\Pi_h\sigma - \sigma_h|_a$ belongs to $\mathbb{K}_h$; we have

$$\begin{aligned}
|\Pi_h\sigma - \sigma_h|_a^2 &= a(\Pi_h\sigma - \sigma, \Pi_h\sigma - \sigma_h) + a(\sigma - \sigma_h, \Pi_h\sigma - \sigma_h)\\
&\le |\Pi_h\sigma - \sigma|_a|\Pi_h\sigma - \sigma_h|_a - b(\Pi_h\sigma - \sigma_h, u - u_h)\\
&= |\Pi_h\sigma - \sigma|_a|\Pi_h\sigma - \sigma_h|_a - b(\Pi_h\sigma - \sigma_h, u)\\
&\le |\Pi_h\sigma - \sigma_h|_a(|\Pi_h\sigma - \sigma|_a + \rho_W(h)\|u\|_{U_0}),
\end{aligned}$$

where $\rho_W(h)$ was introduced in Definition 14.1. This gives

$$\begin{aligned}|\sigma-\sigma_h|_a &\le 2|\Pi_h\sigma-\sigma|_a+\rho_W(h)\|u\|_{U_0}\\ &\le 2\rho_S(h)\|\sigma\|_{\Sigma_0}+\rho_W(h)\|u\|_{U_0},\end{aligned}$$

where $\rho_S(h)$ was introduced in Definition 14.2. The theorem then follows from the definition of the norms of Σ_0 and U_0. □

Before moving to the estimate of the rate of convergence for eigenvalues and eigenfunctions, we present a slight modification of the previous theorem which implies the convergence of T_h to T in $\mathcal{L}(H,U)$.

Theorem 14.7. If there exists a bounded Fortin operator (see Definition 14.4) and if the Fortid condition, the weak approximability of U_0 and the strong approximability of U_0 are satisfied (see Definitions 14.5, 14.1 and 14.2), then there exists $\rho(h)$, tending to zero as h goes to zero, such that

$$\|(T-T_h)g\|_U\le\rho(h)\|g\|_H\quad\forall g\in H.$$

Proof. Let g be in H and consider the solution (σ,u) of (14.2). In particular, we have $u=Tg$. Let us define $u_h=T_hg$ and let σ_h be such that (σ_h,u_h) solves (14.4) (such a σ_h might not be unique). Let $g(h)\in U'$ be such that $\langle g(h),u-u_h\rangle=\|u-u_h\|_U$ and $\|g(h)\|_{U'}=1$. Let $\sigma(h)$ be the first component of the solution to (14.2) with datum $g(h)$, so that $\sigma(h)\in\Sigma_{0'}$ and $\|\sigma(h)\|_{\Sigma_{0'}}\le\|g(h)\|_{U'}=1$. We have

$$\begin{aligned}\|u-u_h\|_U&=\langle g(h),u-u_h\rangle=-b(\sigma(h),u-u_h)\\ &=-b(\sigma(h)-\Pi_h\sigma(h),u-u_h)-b(\Pi_h\sigma(h),u-u_h)\\ &=-b(\sigma(h)-\Pi_h\sigma(h),u-u^I)+a(\sigma-\sigma_h,\Pi_h\sigma(h)).\end{aligned}$$

We estimate the two terms on the right-hand side separately:

$$\begin{aligned}b(\sigma(h)-\Pi_h\sigma(h),u-u^I)&\le C\|\sigma(h)-\Pi_h\sigma(h)\|_\Sigma\|u-u^I\|_U\\ &\le C(\|\sigma(h)\|_\Sigma+\|\Pi_h\sigma(h)\|_\Sigma)\|u-u^I\|_U\end{aligned}$$

and

$$a(\sigma-\sigma_h,\Pi_h\sigma(h))\le|\Pi_h\sigma(h)|_a|\sigma-\sigma_h|_a.$$

The proof is then easily concluded from the definition of the norms in Σ_0, U_0, $\Sigma_{0'}$, and by using the strong approximability to bound $\|u-u^I\|_U$, the boundedness of the Fortin operator and the definition of $\sigma(h)$ to bound $\|\Pi_h\sigma(h)\|_\Sigma$ and $|\Pi_h\sigma(h)|_a$, and the same argument as in the proof of Theorem 14.6 to estimate $|\sigma-\sigma_h|_a$. □

Remark 14.8. The results of Theorems 14.6 and 14.7 can be essentially inverted by stating that suitable norm convergences of T_h to T imply the three main conditions: weak approximability of Definition 14.1, strong approximability of Definition 14.2, and Fortid condition of Definition 14.5. We refer the interested reader to Boffi *et al.* (1997, Theorems 5–7) for the technical details.

We now give basic estimates for the rate of convergence of eigenvalues and eigenfunctions in the spirit of Mercier *et al.* (1981). Let λ be an eigenvalue of (14.1) of multiplicity m and let $E \subset U$ be the corresponding eigenspace. We denote by $\lambda_{1,h}, \lambda_{2,h}, \ldots, \lambda_{m,h}$ the discrete eigenvalues converging to λ and by E_h the direct sum of the corresponding eigenspaces.

The eigenfunction convergence follows directly from the results of the abstract theory presented in Section 6.

Theorem 14.9. Under the hypotheses of Theorem 14.6 or 14.7, there is a constant C such that

$$\hat{\delta}(E, E_h) \leq C\|T - T_h\|_{\mathcal{L}(H,H)},$$

where the gap is evaluated in the H-norm.

Remark 14.10. The same comment as in Remark 13.9 applies to this situation as well. In particular, the convergence of the eigenfunctions in the norm of U would follow from a uniform convergence of T_h to T in $\mathcal{L}(V)$.

In order to estimate the rate of convergence for the eigenvalues, we invoke Mercier *et al.* (1981, Theorem 6.1), where the more general situation of non-symmetric problems is discussed. We assume that the source problems (14.2) and (14.4) are well-posed (with $g \in H$), so that we can define the operator $A : H \to \Sigma$ associated with the first component of the solution as $Ag = \sigma$. Analogously, $A_h : H \to \Sigma_h$ denotes the discrete operator associated to the first component of the solution to problem (14.4): $A_h g = \sigma_h$.

Theorem 14.11. Under the hypotheses of Theorem 14.6 or 14.7 and the additional assumption that the operators A and A_h are well-defined, there exists C such that, for sufficiently small h,

$$\begin{aligned}|\lambda - \lambda_{i,h}| \leq C\big(&\|(T - T_h)_{|E}\|^2_{\mathcal{L}(H,U)} \\ &+ \|(T - T_h)_{|E}\|_{\mathcal{L}(H,U)}\|(A - A_h)_{|E}\|_{\mathcal{L}(H,\Sigma)} \\ &+ \|(A - A_h)_{|E}\|^2_{\mathcal{L}(H,H)}\big),\end{aligned}$$

for $i = 1, \ldots, m$.

We conclude this section with the application of the presented theory to two fundamental examples: the Laplace eigenvalue problem and the biharmonic problem.

14.1. The Laplace problem

Let Ω be an open bounded domain in $\mathbb{R}^n$. The eigenvalue problem associated with the Laplace operator fits within the developed theory with the following definitions:

$$\begin{aligned}
H &= L^2(\Omega),\\
\Sigma &= \mathbf{H}(\mathrm{div};\Omega),\\
U &= L^2(\Omega),\\
a(\boldsymbol{\sigma},\boldsymbol{\tau}) &= \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,\mathrm{d}\mathbf{x},\\
b(\boldsymbol{\tau},v) &= \int_\Omega v\,\mathrm{div}\,\boldsymbol{\tau}\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

It follows, in particular, that the seminorm induced by the form a is indeed the $L^2(\Omega)$-norm,

$$|\cdot|_a = \|\cdot\|_{L^2(\Omega)}.$$

The solution spaces Σ_0 and U_0 contain functions with higher regularity than Σ and U: for a wide class of domains Ω there exists $\varepsilon > 0$ such that $\Sigma_0 \subset H^\varepsilon(\Omega)^n$ and $U_0 \subset H^{1+\varepsilon}(\Omega)$ (if Ω is a convex polygon or polyhedron, the inclusions hold with $\varepsilon = 1$). We should, however, pay attention to the fact that functions $\boldsymbol{\sigma}$ in Σ_0 do not have a more regular divergence than $\mathrm{div}\,\boldsymbol{\sigma} \in L^2(\Omega)$, since from (14.2) we have $\mathrm{div}\,\boldsymbol{\sigma} = -g$, and g varies in $L^2(\Omega)$.

Remark 14.12. The presented setting applies to the Dirichlet problem for Laplace operator. With natural modifications the Neumann problem could be studied as well: $\Sigma = \mathbf{H}_0(\mathrm{div};\Omega)$, $U = L^2_0(\Omega)$.

Several choices of discrete spaces have been proposed for the approximation of Σ and U in two and three space dimensions. In general, U_h is defined as $\mathrm{div}(\Sigma_h)$, where Σ_h is a suitable discretization of $\mathbf{H}(\mathrm{div};\Omega)$. RT (Raviart and Thomas 1977, Nédélec 1980), BDM (Brezzi *et al.* 1985, Brezzi, Douglas and Marini 1986), BDFM (Brezzi *et al.* 1987*b*) elements are possible choices. On quadrilateral meshes, ABF elements (Arnold *et al.* 2005) are a possible solution in order to avoid the lack of convergence arising from the distortion of the elements. In this case the identity $\mathrm{div}(\Sigma_h) = U_h$ is no longer true and the analysis of convergence requires particular care (Gardini 2005).

From the equality $\mathrm{div}(\Sigma_h) = U_h$, it follows that the discrete kernel $\mathbb{K}_h$ contains divergence-free functions, so that the weak approximability (see Definition 14.1) is satisfied:

$$\begin{aligned}
b(\boldsymbol{\tau}_h, v) = b(\boldsymbol{\tau}_h, v - v^I) &\le C\|\boldsymbol{\tau}_h\|_{\mathbf{H}(\mathrm{div};\Omega)}\|v - v^I\|_{L^2(\Omega)}\\
&= C\|\boldsymbol{\tau}_h\|_{L^2(\Omega)}\|v - v^I\|_{L^2(\Omega)},
\end{aligned}$$

for $\boldsymbol{\tau}_h \in \mathbb{K}_h$, $v \in U_0$, and $v^I \in U_h$ suitable approximation of v.

Strong approximability (see Definition 14.2) is a consequence of standard approximation properties in U_0.

It turns out that the main condition for good approximation of the Laplace eigenvalue problem is the Fortid condition (see Definition 14.5). For the schemes mentioned so far (RT, BDM, BDFM), it is not difficult to see that the standard interpolation operator (Brezzi and Fortin 1991) is indeed a Fortin operator (see Definition 14.5): in general, if we denote by $\boldsymbol{\tau}^I$ the interpolant of $\boldsymbol{\tau}$, we have

$$\begin{aligned} b(\boldsymbol{\tau}-\boldsymbol{\tau}^I, v) &= \int_\Omega v \operatorname{div}(\boldsymbol{\tau}-\boldsymbol{\tau}^I)\,\mathrm{d}\mathbf{x} = \sum_K \int_K v \operatorname{div}(\boldsymbol{\tau}-\boldsymbol{\tau}^I)\,\mathrm{d}\mathbf{x} \\ &= -\int_K \mathbf{grad}\, v \cdot (\boldsymbol{\tau}-\boldsymbol{\tau}^I)\,\mathrm{d}\mathbf{x} + \int_{\partial K} v(\boldsymbol{\tau}-\boldsymbol{\tau}^I)\cdot \mathbf{n}\,\mathrm{d}s, \end{aligned}$$

and the degrees of freedom for $\boldsymbol{\tau}^I$ are usually chosen so that the last two terms vanish for all $v \in U_h$. The Fortid condition (see Definition 14.5) is then also a consequence of standard approximation properties. We explicitly notice that a Fortid condition where the term $|\sigma - \Pi_h\sigma|_a$ is replaced by $\|\sigma - \Pi_h\sigma\|_\Sigma$ does not hold in this situation, since this would imply an estimate for $\|\boldsymbol{\tau}-\boldsymbol{\tau}^I\|_{\mathbf{H}(\mathrm{div};\Omega)}$, with $\boldsymbol{\tau} \in \Sigma_0$, but this cannot provide any uniform bound, since $\operatorname{div}\boldsymbol{\tau}$ is a generic element of $L^2(\Omega)$.

We shall return to this example in Section 15.

14.2. The biharmonic problem

For the sake of simplicity, let Ω be a convex polygon in $\mathbb{R}^2$. The biharmonic problem

$$\begin{aligned} \Delta^2 u &= -g \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial \mathbf{n}} &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

has been widely studied in the framework of mixed approximations. In particular, it fits within our setting with the following choices:

$$\begin{aligned} H &= L^2(\Omega), \\ \Sigma &= H^1(\Omega), \\ U &= H^1_0(\Omega), \\ a(\sigma,\tau) &= \int_\Omega \sigma\tau\,\mathrm{d}\mathbf{x}, \\ b(\tau, v) &= -\int_\Omega \mathbf{grad}\,\tau\cdot\mathbf{grad}\,v\,\mathrm{d}\mathbf{x}, \end{aligned}$$

where the auxiliary variable $\sigma = -\Delta u$ has been introduced. It follows that the seminorm associated with the bilinear form a is the $L^2(\Omega)$-norm

$$|\cdot|_a = \|\cdot\|_{L^2(\Omega)}.$$

A possible discretization of the biharmonic problem consists in an equal-order approximation, where Σ_h and U_h are made of continuous piecewise polynomials of degree k (Glowinski 1973, Mercier 1974, Ciarlet and Raviart 1974).

In this case the most delicate condition to be checked is weak approximability (see Definition 14.1). Indeed, strong approximability (see Definition 14.2) is a simple consequence of standard approximation properties. Given $\sigma \in \Sigma_0$, a Fortin operator (14.5) can be defined by $\Pi_h\sigma \in \Sigma_h$ and

$$(\mathbf{grad}\,\Pi_h\sigma, \mathbf{grad}\,\tau) = (\mathbf{grad}\,\sigma, \mathbf{grad}\,\tau) \quad \forall\tau \in \Sigma_h.$$

It is clear that the Fortid condition (see Definition 14.5) holds true.

A direct proof of the weak approximability property requires an inverse estimate (which is valid, for instance, if the mesh is quasi-uniform) and $k \geq 2$:

$$\begin{aligned} b(\tau, v) &= (\mathbf{grad}\,\tau, \mathbf{grad}\,v) = (\mathbf{grad}\,\tau, \mathbf{grad}(v - v^I)) \\ &\leq Ch^{-1}\|\tau\|_{L^2(\Omega)} h^2 \|v\|_{H^3(\Omega)}. \end{aligned}$$

A more refined analysis (Scholz 1978) shows that the weak approximability property is valid for $k = 1$ as well.

15. Inf-sup condition and eigenvalue approximation

In the last section of this part we review the connections between the conditions for good approximation of a source mixed problem (inf-sup conditions) and the conditions for good approximation of the corresponding eigenvalue problem in mixed form (see Sections 13 and 14).

Going back to the notation used at the beginning of Part 3 (the discrete spaces will be denoted by $\Phi_h \subset \Phi$ and $\Xi_h \subset \Xi$), we consider as the two keystones for the approximation of the source problem (12.1) the *ellipticity in the kernel property*: there exists $\alpha > 0$ such that

$$a(\varphi, \varphi) \geq \alpha\|\varphi\|^2_\Phi \quad \forall\varphi \in \mathbb{K}_h, \tag{15.1}$$

where

$$\mathbb{K}_h = \{\varphi \in \Phi_h : b(\varphi, \xi) = 0\ \forall\xi \in \Xi_h\}$$

and the *inf-sup condition*: there exists $\beta > 0$ such that

$$\inf_{\xi\in\Xi_h} \sup_{\varphi_h\in\Phi_h} \frac{b(\varphi, \xi)}{\|\varphi\|_\Phi\|\xi\|_\Xi} \geq \beta \tag{15.2}$$

(Brezzi and Fortin 1991). The ellipticity in the kernel might actually be weakened and written in the form of an inf-sup condition as well, but for our purposes, in the case of symmetric problems, ellipticity in the kernel is a quite general condition.

The conditions presented in Sections 13 and 14 involve quantities other than simply ellipticity in the kernel and the inf-sup condition. Loosely speaking, it turns out that the conditions for good approximation of eigenvalue problems of the first kind require more than ellipticity in the kernel but a weaker inf-sup condition, while the conditions for good approximation of eigenvalue problems of the second kind require the opposite: a stronger inf-sup condition and less than ellipticity in the kernel.

The aim of this section is to review how the conditions introduced in Sections 13 and 14 are actually not equivalent to the ellipticity in the kernel and the inf-sup condition. This is not surprising, for several reasons. First of all, the conditions for the source problem refer to equation (12.1), where the right-hand side consists of the generic functions f and g, which are present in both equations of the mixed problem; in contrast, the eigenvalue problems (13.1) and (14.1) are of different types, since in one equation the right-hand side vanishes. Moreover, there is an intrinsic difference between source problem and eigenvalue problem: the convergence of a source problem is usually reduced to a *pointwise* estimate (*i.e.*, for a generic right-hand side we look for a discrete solution which converges to the continuous one), while the convergence of the eigenvalue problem is related to a *uniform* estimate (see equation (7.8), for instance).

The uniform convergence is usually implied by suitable compactness assumptions and standard pointwise convergence. In the case of a mixed problem, it may be that compactness is not enough to turn pointwise convergence into uniform convergence (see the discussion at the beginning of Part 3 and, in particular, the definition of the solution operator T in (12.2)). A typical example of this situation is the Laplace eigenvalue problem in mixed form, where the operator (12.2) is not compact. It may occur in this situation that ellipticity in the kernel and the inf-sup condition hold true, while the eigenvalues are not correctly approximated. We recall an important counter-example (Boffi *et al.* 2000*a*), already mentioned in Section 5 in the framework of Maxwell's eigenvalue problem (see Table 5.3).

We consider the Laplace eigenvalue problem in mixed form: find $\lambda \in \mathbb{R}$ and $\boldsymbol{\sigma} \in \mathbf{H}(\mathrm{div};\Omega)$ such that, for $u \in L^2(\Omega)$,

$$(\boldsymbol{\sigma},\boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau},u) = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div};\Omega), \tag{15.3a}$$

$$(\mathrm{div}\,\boldsymbol{\sigma},v) = -\lambda(u,v) \quad \forall v \in L^2(\Omega). \tag{15.3b}$$

Its approximation ($\Sigma_h \subset \mathbf{H}(\mathrm{div};\Omega)$, $U_h \subset L^2(\Omega)$) is then as follows: find $\lambda_h \in \mathbb{R}$ and $\boldsymbol{\sigma}_h \in \Sigma_h$ such that, for $u_h \in U_h$,

$$(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau},u_h) = 0 \quad \forall \boldsymbol{\tau} \in \Sigma_h, \tag{15.4a}$$

$$(\mathrm{div}\,\boldsymbol{\sigma}_h,v) = -\lambda_h(u_h,v) \quad \forall v \in U_h. \tag{15.4b}$$

Let us consider a square domain $\Omega =]0,\pi[\times]0,\pi[$ and a criss-cross mesh sequence such as that presented in Figure 3.9. Let Σ_h be the space of continuous piecewise linear finite elements in each component and let U_h be the space containing all the divergences of the elements of Σ_h. It turns out that the equality $U_h = \operatorname{div}(\Sigma_h)$ easily implies that the ellipticity in the kernel property (15.1) is satisfied. Moreover, it can be proved that the proposed scheme satisfies the inf-sup condition (15.2) as well (Boffi *et al.* 2000*a*, Fix, Gunzburger and Nicolaides 1981).

Remark 15.1. Those familiar with the Stokes problem might recognize the inf-sup condition (15.2). It is well known that the $\mathcal{P}_1 - \mathcal{P}_0$ element does not provide a stable Stokes scheme. On the other hand, we are using a very particular mesh sequence (the inf-sup constant would tend to zero on a general mesh sequence) and a norm different to that in the case of the Stokes problem: here we consider the $\mathbf{H}(\operatorname{div};\Omega)$-norm, and not the full $H^1(\Omega)$-norm (even on the criss-cross mesh sequence, the inf-sup constant for the Stokes problem tends to zero when the $H^1(\Omega)$-norm is considered).

The classical theory implies that a quasi-optimal error estimate holds true for the approximation of the source problem associated with problem (15.3). More precisely, we consider the following source problem and its approximation: given $g \in L^2(\Omega)$, find $(\boldsymbol{\sigma}, u) \in \mathbf{H}(\operatorname{div};\Omega) \times L^2(\Omega)$ such that

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, u) = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{H}(\operatorname{div};\Omega), \tag{15.5a}$$

$$(\operatorname{div} \boldsymbol{\sigma}, v) = -(g, v) \quad \forall v \in L^2(\Omega), \tag{15.5b}$$

and find $(\boldsymbol{\sigma}_h, u_h) \in \Sigma_h \times U_h$ such that

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, u_h) = 0 \quad \forall \boldsymbol{\tau} \in \Sigma_h, \tag{15.6a}$$

$$(\operatorname{div} \boldsymbol{\sigma}_h, v) = -(g, v) \quad \forall v \in U_h. \tag{15.6b}$$

Then we have the following error estimate for the solution of problem (15.5):

$$\begin{aligned} &\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}(\operatorname{div};\Omega)} + \|u - u_h\|_{L^2(\Omega)} \\ &\qquad \le C \inf_{\substack{\boldsymbol{\tau}_h \in \Sigma_h \\ v_h \in U_h}} \big(\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{H}(\operatorname{div};\Omega)} + \|u - v_h\|_{L^2(\Omega)} \big). \end{aligned} \tag{15.7}$$

On the other hand, with our choice for the discrete spaces, problem (15.4) does not provide a good approximation of problem (15.3). The results of the computations are shown in Table 15.1. After a transient situation for the smallest meshes, a clear second order of convergence is detected towards the eigenvalues tagged as 'Exact'. Unfortunately, some of them (emphasized by the exclamation mark) do not correspond to eigenvalues of the continuous problem (15.3). This situation is very close to that presented in Table 5.3; it can be observed that the only difference between this computation and that of Section 5 consists in the boundary conditions.

Table 15.1. Eigenvalues computed with nodal elements on the criss-cross mesh sequence of triangles of Figure 3.9.

Exact	Computed (rate)				
	$N=2$	$N=4$	$N=8$	$N=16$	$N=32$
2	2.2606	2.0679 (1.9)	2.0171 (2.0)	2.0043 (2.0)	2.0011 (2.0)
5	4.8634	5.4030 (−1.6)	5.1064 (1.9)	5.0267 (2.0)	5.0067 (2.0)
5	5.6530	5.4030 (0.7)	5.1064 (1.9)	5.0267 (2.0)	5.0067 (2.0)
!→ 6	5.6530	5.6798 (0.1)	5.9230 (2.1)	5.9807 (2.0)	5.9952 (2.0)
8	11.3480	9.0035 (1.7)	8.2715 (1.9)	8.0685 (2.0)	8.0171 (2.0)
10	11.3480	11.3921 (−0.0)	10.4196 (1.7)	10.1067 (2.0)	10.0268 (2.0)
10	12.2376	11.4495 (0.6)	10.4197 (1.8)	10.1067 (2.0)	10.0268 (2.0)
13	12.2376	11.6980 (−0.8)	13.7043 (0.9)	13.1804 (2.0)	13.0452 (2.0)
13	12.9691	11.6980 (−5.4)	13.7043 (0.9)	13.1804 (2.0)	13.0452 (2.0)
!→15	13.9508	15.4308 (1.3)	13.9669 (−1.3)	14.7166 (1.9)	14.9272 (2.0)
!→15	16.1534	15.4308 (1.4)	13.9669 (−1.3)	14.7166 (1.9)	14.9272 (2.0)
17	16.1534	17.0972 (3.1)	18.1841 (−3.6)	17.3073 (1.9)	17.0773 (2.0)
17		18.2042	18.1841 (0.0)	17.3073 (1.9)	17.0773 (2.0)
18		18.3067	19.3208 (−2.1)	18.3456 (1.9)	18.0867 (2.0)
20		20.1735	21.5985 (−3.2)	20.4254 (1.9)	20.1070 (2.0)
20		20.1735	21.6015 (−3.2)	20.4254 (1.9)	20.1070 (2.0)
!→24		27.5131	22.7084 (1.4)	23.6919 (2.1)	23.9230 (2.0)
25		27.6926	24.8559 (4.2)	25.6644 (−2.2)	25.1672 (2.0)
25		28.0122	24.8586 (4.4)	25.6644 (−2.2)	25.1672 (2.0)
26		30.4768	27.3758 (1.7)	26.7152 (0.9)	26.1805 (2.0)

From the computed eigenvalues, it is clear that the conclusions of neither Theorem 14.6 nor 14.7 hold. Let us try to better understand this phenomenon.

Let us forget for a moment the theory developed in Section 14 and try to use the argument of Proposition 7.6, to see whether the pointwise convergence arising from the stability of the source problem approximation implies the required uniform convergence. One of the main issues concerns the appropriate definition of the solution operator.

A first possible (but not useful) definition for the solution operator is given by $T_{\Sigma U} : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$, $T(f,g) = (\sigma, u)$, where (σ, u) is the solution of (15.5) with datum g. We can define the discrete solution operator $T_{\Sigma U,h}$ in a natural way and try to show that the hypotheses of Proposition 7.6 are satisfied. With the notation of Section 7, we have $H = L^2(\Omega)\times L^2(\Omega)$ and $V = \mathbf{H}(\mathrm{div};\Omega)\times L^2(\Omega)$. Unfortunately, it turns out that the operator $T_{\Sigma U}$ is not compact in $\mathcal{L}(L^2(\Omega)\times L^2(\Omega), \mathbf{H}(\mathrm{div};\Omega)\times L^2(\Omega))$ as required; indeed, the component σ of the solution of (15.5) cannot be in a compact subset of $\mathbf{H}(\mathrm{div};\Omega)$ since the divergence of $\boldsymbol{\sigma}$ is equal to g which is only in $L^2(\Omega)$. We can try to use the modified version of Proposition 7.6 introduced in Remark 7.5; however, this does not help since the operator $T_{\Sigma U}$ is not compact in $\mathcal{L}(\mathbf{H}(\mathrm{div};\Omega)\times L^2(\Omega),\mathbf{H}(\mathrm{div};\Omega)\times L^2(\Omega))$ either, for the same reason as before ($\mathrm{div}\,\boldsymbol{\sigma}=-g\in L^2(\Omega)$). On the other hand, the error estimate (15.7) does not give any significant improvement: we have

$$\|(T_{\Sigma U}-T_{\Sigma U,h})(f,g)\|_{\mathbf{H}(\mathrm{div};\Omega)\times L^2(\Omega)} \leq C \inf_{\substack{\boldsymbol{\tau}_h\in\Sigma_h\\ v_h\in U_h}} \big(\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\mathbf{H}(\mathrm{div};\Omega)}+\|u-v_h\|_{L^2(\Omega)}\big).$$

For the same reason as in the previous comments ($\mathrm{div}\,\sigma=-g\in L^2(\Omega)$), we cannot hope to get a uniform bound of the term $\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\mathbf{H}(\mathrm{div};\Omega)}$, for which a higher regularity of $\mathrm{div}\,\boldsymbol{\sigma}$ would be required.

According to Section 14, now let $T: L^2(\Omega)\to L^2(\Omega)$ be the operator associated with the second component of the solution to problem (15.5): $Tg=u$; let $T_h: L^2(\Omega)\to L^2(\Omega)$ be its discrete counterpart, $T_hg=u_h$. A direct application of estimate (15.7) gives, as before,

$$\|(T-T_h)g\|_{L^2(\Omega)} \leq C \inf_{\substack{\boldsymbol{\tau}_h\in\Sigma_h\\ v_h\in U_h}} \big(\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\mathbf{H}(\mathrm{div};\Omega)}+\|u-v_h\|_{L^2(\Omega)}\big).$$

Again, we cannot hope to obtain a uniform convergence since $\mathrm{div}\,\boldsymbol{\sigma}$ is only in $L^2(\Omega)$. Indeed, the profound meaning of Theorems 14.6 and 14.7 in this particular situation is that we try to obtain a uniform estimate by bounding the term $\boldsymbol{\sigma}-\boldsymbol{\sigma}_h$ in $L^2(\Omega)$ and not in $\mathbf{H}(\mathrm{div};\Omega)$; this task is performed by the Fortin operator through the Fortid condition (see Definition 14.5).

A more careful analysis (Boffi *et al.* 2000*a*) of the particular scheme we discuss in this section ($\mathcal{P}_1-\mathcal{P}_0$ element on the criss-cross mesh), shows that, indeed, T_h does not converge uniformly to T.

Proposition 15.2. There exists a sequence $\{v_h^*\}$ with $v_h^*\in U_h$ such that $\|v_h^*\|_{L^2(\Omega)}=1$ for any h and $\|(T-T_h)v_h^*\|_{L^2(\Omega)}$ does not tend to zero as h tends to zero.

Proof. We follow the proof of Boffi *et al.* (2000*a*, Theorem 5.2). Qin (1994), following an idea of Boland and Nicolaides (1985), showed that there exists a sequence $\{\tilde v_h\}$ such that

$$(\mathrm{div}\,\boldsymbol{\tau}_h,\tilde v_h)\leq C\|\boldsymbol{\tau}_h\|_{L^2(\Omega)}\|\tilde v_h\|_{L^2(\Omega)}\quad \forall\boldsymbol{\tau}_h\in\Sigma_h.$$

Defining $v_h^*=\tilde v_h/\|\tilde v_h\|_{L^2(\Omega)}$ gives

$$(\mathrm{div}\,\boldsymbol{\tau}_h,v_h^*)\leq C\|\boldsymbol{\tau}_h\|_{L^2(\Omega)}\quad \forall\boldsymbol{\tau}_h\in\Sigma_h. \tag{15.8}$$

It turns out that v_h^* has zero mean value in each square macro-element composed of four triangles (it is indeed constructed by using a suitable checkerboard structure) and hence its weak limit is zero. From the compactness of T it follows that Tv_h^* tends to zero strongly in $L^2(\Omega)$. By the definition of T_h, we have that $T_h v_h^*$ is the second component u_h of the solution to the following mixed problem:

$$\begin{aligned}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, u_h) &= 0 \qquad \forall \boldsymbol{\tau} \in \Sigma_h,\\ (\operatorname{div} \boldsymbol{\sigma}_h, v) &= -(v_h^*, v) \qquad \forall v \in U_h.\end{aligned}$$

We have

$$\begin{aligned}|(\operatorname{div} \boldsymbol{\sigma}_h, u_h)| &= |(v_h^*, u_h)| \leq \|u_h\|_{L^2(\Omega)},\\ |(\operatorname{div} \boldsymbol{\sigma}_h, u_h)| &= \|\boldsymbol{\sigma}_h\|^2_{L^2(\Omega)}.\end{aligned}$$

Moreover, from (15.8) we have

$$\|\boldsymbol{\sigma}_h\|_{L^2(\Omega)} \geq \frac{1}{C} |(\operatorname{div} \boldsymbol{\sigma}_h, v_h^*)| = \frac{1}{C}.$$

Putting together the last equations gives the final result,

$$\|u_h\|_{L^2(\Omega)} \geq \frac{1}{C^2}. \qquad \square$$

PART FOUR
The language of differential forms

The use of differential forms and homological techniques for the analysis of the finite element approximation of partial differential equations has become a popular and effective tool in the recent literature.

The aim of this part is to provide the reader with some basic notions about the de Rham complex and its role in the analysis of eigenvalue problems arising from partial differential equations. The experience of the author in this field comes from the approximation of Maxwell's eigenvalue problem; we shall, however, see how the abstract setting can be used for the analysis of a wider class of applications. For a thorough introduction to this subject in the framework of the numerical analysis of partial differential equations, we refer the interested reader to Arnold, Falk and Winther (2006*b*).

16. Preliminaries

In this section we recall the definitions of the main entities we are going to use in our analysis.

For any integer k with $0 \le k \le n$, let $\Lambda^k = \Lambda^k(\mathbb{R}^n)$ be the vector space of alternating k-linear forms on $\mathbb{R}^n$, so $\dim(\Lambda^k) = \binom{n}{k}$. The Euclidean norm of $\mathbb{R}^n$ induces a norm on Λ^k for any k and the wedge product '$\wedge$' acts from $\Lambda^i \times \Lambda^j$ to Λ^{i+j}. Given a domain $\Omega \subset \mathbb{R}^n$, we denote by $C^\infty(\Omega, \Lambda^k)$ the space of smooth differential forms of order k in Ω and by $\mathbf{\Lambda}(\Omega)$ the corresponding anti-commutative graded algebra

$$\mathbf{\Lambda}(\Omega) = \bigoplus_k C^\infty(\Omega, \Lambda^k).$$

An exterior derivative $\mathbf{d} : \mathbf{\Lambda}(\Omega) \to \mathbf{\Lambda}(\Omega)$ is a linear graded operator of degree one, that is, for any k, we are given a map

$$d_k : C^\infty(\Omega, \Lambda^k) \to C^\infty(\Omega, \Lambda^{k+1}).$$

We assume that $d_{k+1} \circ d_k = 0$, that is, $\mathbf{d}$ is a differential.[1]

The set $\mathbf{L}^2(\Omega, \Lambda^k)$ is the space of differential k-forms on Ω with square integrable coefficients in their canonical basis representation; its inner product is given by

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \wedge \star\mathbf{v} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega, \Lambda^k),$$

[1] The external derivative we are going to use can be naturally defined as follows: given $\omega \in C^\infty(\Omega, \Lambda^k)$, $d_k\omega_x(v_1, \dots, v_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \partial_{v_j} \omega_x(v_1, \dots, \hat{v}_j, \dots, v_{k+1})$ (Arnold *et al.* 2006*b*, p. 15)

where $\star$ denotes the Hodge star operator mapping k-forms to $(n-k)$-forms. Differential k-forms with different regularity can be considered by using standard Sobolev spaces: the corresponding spaces will be denoted by $\mathbf{H}^s(\Omega, \Lambda^k)$. The main spaces of our construction are based on the exterior derivative

$$\mathbf{H}(d_k; \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega, \Lambda^k) : d_k \mathbf{v} \in \mathbf{L}^2(\Omega, \Lambda^{k+1})\},$$

and are endowed with their natural scalar product

$$(\mathbf{u}, \mathbf{v})^2_{\mathbf{H}(d_k;\Omega)} = (\mathbf{u}, \mathbf{v})^2_{\mathbf{L}^2(\Omega,\Lambda^k)} + (d_k\mathbf{u}, d_k\mathbf{v})^2_{\mathbf{L}^2(\Omega,\Lambda^{k+1})} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}(d_k; \Omega).$$

It can be shown that a trace operator $\mathbf{tr}_{\partial\Omega}$ is well defined on $\mathbf{H}(d_k; \Omega)$, so that it makes sense to introduce the subspace of $\mathbf{H}(d_k; \Omega)$ consisting of differential forms with vanishing boundary conditions:

$$\mathbf{H}_0(d_k; \Omega) = \{\mathbf{v} \in \mathbf{H}(d_k; \Omega) : \mathbf{tr}_{\partial\Omega}\, \mathbf{v} = 0\}.$$

The coderivative operator $\delta_k : C^\infty(\Omega, \Lambda^k) \to C^\infty(\Omega, \Lambda^{k-1})$ is defined by

$$\delta_k = \star d_{n-k} \star,$$

and is the formal adjoint of d_{k-1}. Indeed, we have the following generalization of the integration by parts:

$$(d_{k-1}\mathbf{p}, \mathbf{u}) = (\mathbf{p}, \delta_k \mathbf{u}) + \langle \mathbf{tr}_{\partial\Omega}\, \mathbf{p}, \mathbf{tr}_{\partial\Omega} \star \mathbf{u} \rangle.$$

We can define Hilbert spaces associated with the coderivative

$$\mathbf{H}(\delta_k; \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega, \Lambda^k) : \delta_k \mathbf{v} \in \mathbf{L}^2(\Omega, \Lambda^{k-1})\}.$$

For $\mathbf{u} \in \mathbf{H}(\delta_k; \Omega)$ we have $\star\mathbf{u} \in \mathbf{H}(d_{n-k}; \Omega)$, so it makes sense to consider $\mathbf{tr}_{\partial\Omega}(\star\mathbf{u})$ and to define $\mathbf{H}_0(\delta_k; \Omega)$ as $\star\mathbf{H}_0(d_{n-k}; \Omega)$, that is,

$$\mathbf{H}_0(\delta_k; \Omega) = \{\mathbf{v} \in \mathbf{H}(\delta_k; \Omega) : \mathbf{tr}_{\partial\Omega}(\star\mathbf{v}) = 0\}.$$

Before introducing additional definitions and the fundamental notion of the de Rham complex, it might be useful to recall how functions of differential forms can be identified with standard functional spaces in two and three space dimensions. Following Arnold *et al.* (2006*b*), the identification is performed in a standard way by means of Euclidean vector proxies and is reported in Table 16.1 (Boffi *et al.* 2009).

The de Rham complex is given by the chain

$$0 \longrightarrow \mathbf{H}(d_0; \Omega) \xrightarrow{d_0} \mathbf{H}(d_1; \Omega) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \mathbf{H}(d_n; \Omega) \longrightarrow 0.$$

The analogous complex when boundary conditions are considered is

$$0 \longrightarrow \mathbf{H}_0(d_0; \Omega) \xrightarrow{d_0} \mathbf{H}_0(d_1; \Omega) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \mathbf{H}_0(d_n; \Omega) \longrightarrow 0.$$

Table 16.1. Identification between differential forms and vector proxies in $\mathbb{R}^2$ and $\mathbb{R}^3$.

	Differential form	Proxy representation $n=2$	$n=3$
$k=0$	d_0	**grad**	**grad**
	$\mathbf{tr}_{\partial\Omega}\,\phi$	$\phi_{\vert\partial\Omega}$	$\phi_{\vert\partial\Omega}$
	$H_0(d_0,\Omega)$	$H^1_0(\Omega)$	$H^1_0(\Omega)$
	δ_1	$-\operatorname{div}$	$-\operatorname{div}$
$k=1$	d_1	rot	**curl**
	$\mathbf{tr}_{\partial\Omega}\,\mathbf{u}$	$(\mathbf{u}\cdot\mathbf{t})_{\vert\partial\Omega}$	$\mathbf{n}\times(\mathbf{u}\times\mathbf{n})_{\vert\partial\Omega}$
	$H_0(d_1,\Omega)$	$\mathbf{H}_0(\mathrm{rot})$	$\mathbf{H}_0(\mathbf{curl})$
	δ_2	$\overrightarrow{\mathrm{rot}}$	**curl**
$k=2$	d_2	0	div
	$\mathbf{tr}_{\partial\Omega}\,\mathbf{q}$	0	$(\mathbf{q}\cdot\mathbf{n})_{\vert\partial\Omega}$
	$H_0(d_2,\Omega)$	$L^2_0(\Omega)$	$\mathbf{H}_0(\mathrm{div})$
	δ_3		$-\,$**grad**

Since $d_{k+1}\circ d_k = 0$, the de Rham complex is a cochain complex, that is, the kernel of d_{k+1} contains the range of d_k. The quotient spaces between the kernels and the ranges of the exterior derivatives are called cohomology spaces and have finite dimension, which is related to the topology of Ω (the dimension of the kth cohomology space is called kth Betti number). The kth cohomology space is given by the set of harmonic differential forms:

$$\boldsymbol{\mathcal{H}}^k = \{\mathbf{v}\in\mathbf{H}(d_k;\Omega)\cap\mathbf{H}_0(\delta_k;\Omega) : d_k\mathbf{v}=0,\ \delta_k\mathbf{v}=0\}$$

and

$$\boldsymbol{\mathcal{H}}^k_0 = \{\mathbf{v}\in\mathbf{H}_0(d_k;\Omega)\cap\mathbf{H}(\delta_k;\Omega) : d_k\mathbf{v}=0,\ \delta_k\mathbf{v}=0\},$$

respectively.

In this survey, we are going to consider the case when the Betti numbers corresponding to k different from 0 and n vanish. This essentially means that the de Rham complex is exact[2] and corresponds to the case when the domain Ω is contractible. On the other hand, even if this assumption

[2] The de Rham complex is exact in the case of trivial cohomology if, in the definition of the first (last, respectively, when boundary conditions are considered) space, 0 is replaced by $\mathbb{R}$. We consider this modification in the rest of our survey.

considerably simplifies the exposition, the results we are going to present generalize to the case when the topology may not be trivial; the techniques for dealing with the more general case correspond to those used, for instance, in Arnold *et al.* (2006*b*).

For the discretization of the spaces of differential forms, we introduce spaces of discrete differential forms $\boldsymbol{\mathcal{V}}_h^k \subset \mathbf{H}(d_k;\Omega)$ ($k=0,\dots,n$). A typical setting involves appropriate projection operators $\pi_h^k : \mathbf{H}(d_k;\Omega) \to \boldsymbol{\mathcal{V}}_h^k$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & \mathbf{H}(d_0;\Omega) & \xrightarrow{d_0} & \mathbf{H}(d_1;\Omega) & \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} & \mathbf{H}(d_n;\Omega) & \longrightarrow & 0\\
 & & \pi_h^0\big\downarrow & & \pi_h^1\big\downarrow & & \pi_h^n\big\downarrow & & \\
\mathbb{R} & \longrightarrow & \boldsymbol{\mathcal{V}}_h^0 & \xrightarrow{d_0} & \boldsymbol{\mathcal{V}}_h^1 & \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} & \boldsymbol{\mathcal{V}}_h^n & \longrightarrow & 0.
\end{array}$$

We have an analogous diagram when boundary conditions are considered. If $\boldsymbol{\mathcal{V}}_h^k \subset \mathbf{H}_0(d_k;\Omega)$ ($k=0,\dots,n$) and suitable projection operators $\pi_h^k : \mathbf{H}_0(d_k;\Omega) \to \boldsymbol{\mathcal{V}}_h^k$ are considered, then we use

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{H}_0(d_0;\Omega) & \xrightarrow{d_0} & \mathbf{H}_0(d_1;\Omega) & \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} & \mathbf{H}_0(d_n;\Omega) & \longrightarrow & \mathbb{R}\\
 & & \pi_h^0\big\downarrow & & \pi_h^1\big\downarrow & & \pi_h^n\big\downarrow & & \\
0 & \longrightarrow & \boldsymbol{\mathcal{V}}_h^0 & \xrightarrow{d_0} & \boldsymbol{\mathcal{V}}_h^1 & \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} & \boldsymbol{\mathcal{V}}_h^n & \longrightarrow & \mathbb{R}.
\end{array} \tag{16.1}$$

Another important tool we use is the Hodge decomposition, which can be easily expressed by means of the cycles and the boundaries coming from the de Rham complex (Arnold *et al.* 2006*b*, equation 2.18). Roughly speaking, the Hodge decomposition states that every k-form $\mathbf{u}$ can be split as the sum of three components:

$$\mathbf{u} = d_{k-1}\boldsymbol{\alpha} + \delta_{k+1}\boldsymbol{\beta} + \boldsymbol{\gamma},$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $(k-1)$- and $(k+1)$-forms, respectively, and $\boldsymbol{\gamma} \in \boldsymbol{\mathcal{H}}^k$ is a harmonic k-form. More precisely, it turns out that exact (*i.e.*, in the range of d_{k-1}) and co-exact (*i.e.*, in the range of δ_{k+1}) k-forms are orthogonal in $\mathbf{L}^2(\Omega,\Lambda^k)$: it follows that the orthogonal complement of exact and co-exact k-forms consists of forms that are simultaneously closed (*i.e.*, in the kernel of d_k) and co-closed (*i.e.*, in the kernel of δ_k), that is, of harmonic k-forms. In the particular case we consider, there are no harmonic forms and the Hodge decomposition says that $\mathbf{L}^2(\Omega,\Lambda^k)$ can be presented as the direct sum of $d_{k-1}(\mathbf{H}(d_{k-1};\Omega))$ and $\delta_{k+1}(\mathbf{H}_0(\delta_{k+1};\Omega))$. A second Hodge decomposition with different boundary conditions says

$$\mathbf{L}^2(\Omega,\Lambda^k) = d_{k-1}(\mathbf{H}_0(d_{k-1};\Omega)) \oplus \delta_{k+1}(\mathbf{H}(\delta_{k+1};\Omega)). \tag{16.2}$$

17. The Hodge–Laplace eigenvalue problem

The main object of our analysis is the following symmetric variational eigenvalue problem: find $\lambda \in \mathbb{R}$ and $\mathbf{u} \in \mathbf{H}_0(d_k;\Omega)$ with $\mathbf{u} \neq 0$ such that

$$(d_k\mathbf{u}, d_k\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(d_k;\Omega). \tag{17.1}$$

Taking $\mathbf{v} = \mathbf{u}$, it follows that λ cannot be negative. For $k = 0$, problem (17.1) reduces to the standard Laplace eigenvalue problem. For $k \geq 1$, which is the most interesting case, $\lambda = 0$ implies $d_k\mathbf{u} = 0$, that is, $\mathbf{u} = d_{k-1}\boldsymbol{\alpha}$ for some $\boldsymbol{\alpha} \in \mathbf{H}_0(d_{k-1};\Omega)$. On the other hand, $\lambda > 0$ implies $\delta_k\mathbf{u} = 0$, as can be seen by taking $\mathbf{v} = d_{k-1}\boldsymbol{\alpha}$ in (17.1) for an arbitrary $\boldsymbol{\alpha} \in \mathbf{H}_0(d_{k-1};\Omega)$ and using the orthogonalities discussed when introducing the Hodge decomposition (16.2). This means that intrinsic constraints are hidden in the formulation of problem (17.1): either λ vanishes and is associated to the infinite-dimensional eigenspace $d_{k-1}(\mathbf{H}_0(d_{k-1};\Omega))$ (that is, $\mathbf{u}$ is a closed form), or the eigenfunctions $\mathbf{u}$ associated with positive values of λ are coclosed forms (that is, $\delta_k\mathbf{u} = 0$).

One motivation for the study of problem (17.1) comes from the fact that, for $k = 1$ and $n = 3$, it corresponds to the Maxwell eigenvalue problem (5.3) (see the identifications in Table 16.1). Moreover, for $k = 0$, problem (17.1) reduces to the well-known eigenvalue problem for the Laplace operator. Another interesting application is given for $k = 2$ and $n = 3$, where the eigenvalue problem associated with the **grad** div operator is obtained: this operator plays an important role in the approximation of fluid–structure interaction and acoustic problems (Bermúdez and Rodríguez 1994, Bermúdez *et al.* 1995, Bathe, Nitikitpaiboon and Wang 1995, Gastaldi 1996, Boffi, Chinosi and Gastaldi 2000*b*).

Remark 17.1. We have introduced the eigenvalue problem (17.1) in the space $\mathbf{H}_0(d_k;\Omega)$, which involves essential Dirichlet boundary conditions in the sense that the definition of our functional space implies $\mathbf{tr}_{\partial\Omega}\,\mathbf{u} = 0$. The same problem can be considered in the space $\mathbf{H}(d_k;\Omega)$ and would correspond to natural Neumann boundary conditions. Since the modifications involved with the analysis of the Neumann problem are standard, we limit our presentation to the Dirichlet problem.

Remark 17.2. The term 'Dirichlet' used in the previous remark needs a more precise explanation. From the technical point of view, problem (17.1) is a Dirichlet problem, since essential boundary conditions are imposed in the space $\mathbf{H}_0(d_k;\Omega)$ and we are sticking with this terminology.

On the other hand, a Dirichlet problem in the framework of differential forms might correspond to a different type of boundary conditions when translated to a more conventional language. This is sometimes the case when using the proxy identification of Table 16.1 to reduce problem (17.1) to standard mixed formulations. For instance, the case $k = 2$ and $n = 3$

corresponds to an eigenvalue problem associated with the **grad** div operator, which turns out to be equivalent to the Neumann problem for the Laplace eigenvalue problem. A similar situation occurs in the case $k = 1$ and $n = 2$, which corresponds to the Maxwell eigenvalue problem in two space dimensions with perfectly conducting boundary conditions $\mathbf{u} \cdot \mathbf{t} = 0$ on the boundary. This problem is equivalent to the mixed formulation of the Neumann–Laplace eigenvalue problem ($\mathbf{u}$ corresponds to the gradient of the solution u rotated by the angle $\pi/2$, so that $\mathbf{u} \cdot \mathbf{t}$ means $\partial u/\partial n$).

Problem (17.1) is strictly related to the Hodge–Laplace elliptic eigenvalue problem (Arnold *et al.* 2006*b*, Arnold, Falk and Winther 2010): find $\omega \in \mathbb{R}$ and $\boldsymbol{\sigma} \in \mathbf{H}_0(d_k;\Omega) \cap \mathbf{H}(\delta_k;\Omega)$ with $\boldsymbol{\sigma} \neq 0$ such that

$$(d_k\boldsymbol{\sigma}, d_k\boldsymbol{\tau}) + (\delta_k\boldsymbol{\sigma}, \delta_k\boldsymbol{\tau}) = \omega(\boldsymbol{\sigma}, \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \mathbf{H}_0(d_k;\Omega) \cap \mathbf{H}(\delta_k;\Omega). \tag{17.2}$$

It is clear that all solutions to (17.2) have positive frequency $\omega > 0$ (since Ω is assumed to have trivial topology). Moreover, problem (17.2) is associated with a *compact* solution operator; this important property is a consequence of the compact embedding of $\mathbf{H}_0(d_k;\Omega) \cap \mathbf{H}(\delta_k;\Omega)$ into $\mathbf{L}^2(\Omega, \Lambda^k)$ (Picard 1984).

It is possible to classify the solutions to (17.2) into two distinct families. The first family corresponds to the solutions $(\lambda, \mathbf{u})$ to (17.1) with positive frequency (we have already observed that in this case $\delta_k\mathbf{u} = 0$, so that it is clear that $(\omega, \boldsymbol{\sigma}) = (\lambda, \mathbf{u})$ is a solution to (17.2)). The second family is given by forms $\boldsymbol{\sigma} \in \mathbf{H}(\delta_k;\Omega)$ satisfying

$$(\delta_k\boldsymbol{\sigma}, \delta_k\boldsymbol{\tau}) = \omega(\boldsymbol{\sigma}, \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \mathbf{H}(\delta_k;\Omega)$$

with $\omega > 0$, which implies, in particular, $d_k\boldsymbol{\sigma} = 0$.

We are interested in the first family of solutions to the Hodge–Laplace eigenvalue problem (17.2) in the case $k \geq 1$. From the above discussion, the problem can be written in the following way: find $\lambda \in \mathbb{R}$ and $\mathbf{u} \in \mathbf{H}_0(d_k;\Omega)$ with $\mathbf{u} \neq 0$ such that

$$(d_k\mathbf{u}, d_k\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(d_k;\Omega), \tag{17.3a}$$

$$(\mathbf{u}, d_{k-1}\mathbf{q}) = 0 \quad \forall \mathbf{q} \in \mathbf{H}_0(d_{k-1};\Omega). \tag{17.3b}$$

A natural mixed formulation associated with problem (17.3) can be constructed as follows: find $\lambda \in \mathbb{R}$ and $\mathbf{u} \in \mathbf{H}_0(d_k;\Omega)$ with $\mathbf{u} \neq 0$ such that, for $\mathbf{p} \in \mathbf{H}_0(d_{k-1};\Omega)$,

$$(d_k\mathbf{u}, d_k\mathbf{v}) + (\mathbf{v}, d_{k-1}\mathbf{p}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(d_k;\Omega), \tag{17.4a}$$

$$(\mathbf{u}, d_{k-1}\mathbf{q}) = 0 \quad \forall \mathbf{q} \in \mathbf{H}_0(d_{k-1};\Omega). \tag{17.4b}$$

Taking $\mathbf{v} = d_{k-1}\mathbf{p}$ in (17.4a) and using (17.4b) easily give $d_{k-1}\mathbf{p} = 0$, which shows that all solutions to (17.4) solve (17.3) as well. *Vice versa*, it is clear that a solution to (17.3) solves (17.4) with $\mathbf{p} = 0$.

In the case of Maxwell's eigenvalue problem, formulation (17.4) is often referred to as Kikuchi's formulation (Kikuchi 1987).

It is clear that the value of $\mathbf{p}$ cannot be uniquely determined when $k \geq 2$, since if $(\lambda, \mathbf{u}, \mathbf{p})$ satisfies (17.4), then $(\lambda, \mathbf{u}, \mathbf{p} + d_{k-2}\boldsymbol{\alpha})$ satisfies (17.4) as well for any $\boldsymbol{\alpha} \in \mathbf{H}_0(d_{k-2}; \Omega)$. It might be then interesting to consider the following modified problem, which avoids the indeterminacy of $\mathbf{p}$. Given the space

$$\mathbf{K}^\delta_{k-1} = \{\mathbf{v} \in \mathbf{H}_0(d_{k-1}; \Omega) \cap \mathbf{H}(\delta_{k-1}; \Omega) : \delta_{k-1}\mathbf{v} = 0\},$$

find $\lambda \in \mathbb{R}$ and $\mathbf{u} \in \mathbf{H}_0(d_k; \Omega)$ with $\mathbf{u} \neq 0$ such that, for $\mathbf{p} \in \mathbf{K}^\delta_{k-1}$,

$$(d_k\mathbf{u}, d_k\mathbf{v}) + (\mathbf{v}, d_{k-1}\mathbf{p}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(d_k; \Omega), \tag{17.5a}$$

$$(\mathbf{u}, d_{k-1}, \mathbf{q}) = 0 \quad \forall \mathbf{q} \in \mathbf{K}^\delta_{k-1}. \tag{17.5b}$$

Formulation (17.5) is, however, not suited to the numerical approximation, since it is not obvious how to introduce a conforming approximation of the space $\mathbf{K}^\delta_{k-1}$. For this reason, we are going to use formulation (17.4); the fact that $\mathbf{p}$ might not be uniquely determined by $\mathbf{u}$ is not a problem, since we are interested in the eigenfunction $\mathbf{u}$.

Following Boffi *et al.* (1999*b*), a second mixed formulation can be obtained as follows. Given the space

$$\mathbf{K}^d_{k+1} = d_k(\mathbf{H}_0(d_k; \Omega)) \subset \mathbf{H}_0(d_{k+1}; \Omega),$$

find $\lambda \in \mathbb{R}$ and $\mathbf{s} \in \mathbf{K}^d_{k+1}$ such that, for $\mathbf{u} \in \mathbf{H}_0(d_k; \Omega)$,

$$(\mathbf{u}, \mathbf{v}) + (d_k\mathbf{v}, \mathbf{s}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0(d_k; \Omega), \tag{17.6a}$$

$$(d_k\mathbf{u}, \mathbf{t}) = -\lambda(\mathbf{s}, \mathbf{t}) \quad \forall \mathbf{t} \in \mathbf{K}^d_{k+1}. \tag{17.6b}$$

It is clear that any solution to problem (17.6) is associated with a positive frequency $\lambda > 0$. Indeed, if $\lambda = 0$ then (17.6b) implies $d_k\mathbf{u} = 0$, and taking $\mathbf{v} = \mathbf{u}$ in (17.6a) gives $\mathbf{u} = 0$, which contradicts the existence of solutions with vanishing frequency. Hence, (17.6b) gives the fundamental relation $\mathbf{s} = -d_k\mathbf{u}/\lambda$, which yields the equivalence between (17.6a) and (17.3a).

Remark 17.3. The similar notations used for the spaces $\mathbf{K}^\delta_{k-1}$ and $\mathbf{K}^d_{k+1}$ are compatible, in the sense that the space $\mathbf{K}^d_{k+1}$ is made of functions in the kernel of the operator d_{k+1}, in analogy to the space $\mathbf{K}^\delta_{k-1}$, which contains functions in the kernel of the operator δ_{k-1}.

The equivalence between the mixed formulations and problem (17.1) is stated in the following proposition.

Proposition 17.4. If $(\lambda, \mathbf{u}) \in \mathbb{R} \times \mathbf{H}_0(d_k; \Omega)$ is a solution of (17.1) with $\lambda > 0$, that is, if $(\lambda, \mathbf{u})$ is a solution of (17.3), then $(\lambda, \mathbf{u})$ is a solution of (17.4), and there exists $\mathbf{s} \in \mathbf{K}^d_{k+1}$ such that $(\lambda, \mathbf{s})$ is a solution of (17.6).

Conversely, if $(\lambda, \mathbf{u}) \in \mathbb{R} \times \mathbf{H}_0(d_k;\Omega)$ is a solution of (17.4), then $\lambda > 0$ and $(\lambda, \mathbf{u})$ solves (17.3) and (17.1); if $(\lambda, \mathbf{s}) \in \mathbb{R} \times \mathbf{K}^d_{k+1}$ is a solution of (17.6) for some $\mathbf{u} \in \mathbf{H}_0(d_k;\Omega)$, then $(\lambda, \mathbf{u})$ solves (17.3) and (17.1).

18. Approximation of the mixed formulations

The aim of this section is to translate the abstract theory presented in Part 3 into the language of differential forms and to apply it to the approximation of problems (17.4) and (17.6).

18.1. Approximation of problem (17.4)

With the notation introduced at the beginning of this part, the discretization of problem (17.4) involves the spaces $\boldsymbol{\mathcal{V}}_h^k \subset \mathbf{H}_0(d_k;\Omega)$ and $\boldsymbol{\mathcal{V}}_h^{k-1} \subset \mathbf{H}_0(d_{k-1};\Omega)$. The discrete formulation is as follows: find $\lambda_h \in \mathbb{R}$ and $\mathbf{u}_h \in \boldsymbol{\mathcal{V}}_h^k$ with $\mathbf{u}_h \neq 0$ such that, for $\mathbf{p}_h \in \boldsymbol{\mathcal{V}}_h^{k-1}$,

$$(d_k\mathbf{u}_h, d_k\mathbf{v}) + (\mathbf{v}, d_{k-1}\mathbf{p}_h) = \lambda_h(\mathbf{u}_h, \mathbf{v}) \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{V}}_h^k, \tag{18.1a}$$

$$(\mathbf{u}_h, d_{k-1}\mathbf{q}) = 0 \quad \forall \mathbf{q} \in \boldsymbol{\mathcal{V}}_h^{k-1}. \tag{18.1b}$$

This is a mixed problem of the first kind, so we can analyse it with the tools introduced in Section 13. According to the discussion of Section 13, we can define a continuous operator $T^{(1)} : \mathbf{L}^2(\Omega, \Lambda^k) \to \mathbf{H}_0(d_k;\Omega)$ and a discrete operator $T_h^{(1)} : \mathbf{L}^2(\Omega, \Lambda^k) \to \boldsymbol{\mathcal{V}}_h^k$ related to the first component of the solution of the corresponding (continuous and discrete) source problems, which we now write explicitly for the reader's convenience. The continuous source problem is as follows: given $\mathbf{f} \in \mathbf{L}^2(\Omega, \Lambda^k)$, find $\mathbf{u} \in \mathbf{H}_0(d_k;\Omega)$ and $\mathbf{p} \in \mathbf{H}_0(d_{k-1};\Omega)$ such that

$$(d_k\mathbf{u}, d_k\mathbf{v}) + (\mathbf{v}, d_{k-1}\mathbf{p}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(d_k;\Omega),$$

$$(\mathbf{u}, d_{k-1}\mathbf{q}) = 0 \quad \forall \mathbf{q} \in \mathbf{H}_0(d_{k-1};\Omega),$$

and its discrete counterpart is to find $\mathbf{u}_h \in \boldsymbol{\mathcal{V}}_h^k$ and $\mathbf{p}_h \in \boldsymbol{\mathcal{V}}_h^{k-1}$ such that

$$(d_k\mathbf{u}_h, d_k\mathbf{v}) + (\mathbf{v}, d_{k-1}\mathbf{p}_h) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{V}}_h^k,$$

$$(\mathbf{u}_h, d_{k-1}\mathbf{q}) = 0 \quad \forall \mathbf{q} \in \boldsymbol{\mathcal{V}}_h^{k-1}.$$

To apply the theory developed in Section 13, we need to show that

$$T^{(1)} \text{ is compact from } \mathbf{L}^2(\Omega, \Lambda^k) \text{ to } \mathbf{H}_0(d_k;\Omega).$$

From compactness of the embedding of $\mathbf{H}_0(d_k;\Omega) \cap \mathbf{H}(\delta_k;\Omega)$ into $\mathbf{L}^2(\Omega, \Lambda^k)$, it follows that $T^{(1)}$ is a compact operator from $\mathbf{L}^2(\Omega, \Lambda^k)$ into $\mathbf{L}^2(\Omega, \Lambda^k)$. Moreover, it turns out that $d_k(T^{(1)}\mathbf{L}^2(\Omega, \Lambda^k))$ is contained in $\mathbf{H}_0(d_{k+1};\Omega) \cap \mathbf{H}(\delta_{k+1};\Omega)$, which is compactly embedded into $\mathbf{L}^2(\Omega, \Lambda^{k+1})$. This implies that $T^{(1)}$ has the required compactness from $\mathbf{L}^2(\Omega, \Lambda^k)$ into $\mathbf{H}_0(d_k;\Omega)$.

In agreement with Section 13, we introduce some spaces. Let $\mathbb{K}$ be the kernel of the operator δ_k, that is,

$$\mathbb{K} = \{\mathbf{v} \in \mathbf{H}_0(d_k;\Omega) : (\mathbf{v}, d_{k-1}\mathbf{q}) = 0\ \forall \mathbf{q} \in \mathbf{H}_0(d_{k-1};\Omega)\}$$

and its discrete counterpart

$$\mathbb{K}_h = \{\mathbf{v} \in \boldsymbol{\mathcal{V}}_h^k : (\mathbf{v}, d_{k-1}\mathbf{q}) = 0\ \forall \mathbf{q} \in \boldsymbol{\mathcal{V}}_h^{k-1}\}.$$

Moreover, $\mathcal{V}_0$ and $\mathcal{Q}_0$ denote the spaces containing all solutions $\mathbf{u}$ and $\mathbf{p}$, respectively, of the continuous source problem (remember that the component $\mathbf{p}$ of the solution might not be unique).

The three fundamental hypotheses of Theorem 13.4 are the *ellipticity in the kernel*, the *weak approximability* of $\mathcal{Q}_0$ and the *strong approximability* of $\mathcal{V}_0$ (see Definitions 13.1, 13.2, and 13.3). For the reader's convenience, we recall these properties with the actual notation.

The ellipticity in the kernel states that there exists $\alpha > 0$ such that

$$(d_k\mathbf{v}, d_k\mathbf{v}) \geq \alpha(\mathbf{v}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{K}_h. \tag{18.2}$$

The weak approximability of $\mathcal{Q}_0$ means that there exists $\rho_W(h)$, tending to zero as h tends to zero, such that

$$\sup_{\mathbf{v}\in\mathbb{K}_h} \frac{(\mathbf{v}, d_{k-1}\mathbf{q})}{\|\mathbf{v}\|_{\mathbf{H}(d_k;\Omega)}} \leq \rho_W(h)\|\mathbf{q}\|_{\mathcal{Q}_0}. \tag{18.3}$$

The strong approximability of $\mathcal{V}_0$ means that there exists $\rho_S(h)$, tending to zero as h tends to zero, such that, for any $\mathbf{u} \in \mathcal{V}_0$, there exists $\mathbf{u}^I \in \mathbb{K}_h$ with

$$\|\mathbf{u} - \mathbf{u}^I\|_{\mathbf{H}(d_k;\Omega)} \leq \rho_S(h)\|\mathbf{u}\|_{\mathcal{V}_0}. \tag{18.4}$$

The next proposition is the analogue of Theorem 13.4 in the setting of this section.

Proposition 18.1. If the ellipticity in the kernel (18.2), the weak approximability of $\mathcal{Q}_0$ (18.3), and the strong approximability of $\mathcal{V}_0$ (18.4) hold true, then there exists $\rho(h)$, tending to zero as h tends to zero, such that

$$\|(T^{(1)} - T_h^{(1)})\mathbf{f}\|_{\mathbf{H}(d_k;\Omega)} \leq \rho(h)\|\mathbf{f}\|_{\mathbf{L}^2(\Omega,\Lambda^k)} \quad \forall \mathbf{f} \in \mathbf{L}^2(\Omega, \Lambda^k).$$

18.2. Approximation of problem (17.6)

The approximation of the second mixed formulation (17.6) reads as follows: find $\lambda_h \in \mathbb{R}$ and $\mathbf{s}_h \in d_k(\boldsymbol{\mathcal{V}}_h^k)$ such that, for $\mathbf{u}_h \in \boldsymbol{\mathcal{V}}_h^k$,

$$(\mathbf{u}_h, \mathbf{v}) + (d_k\mathbf{v}, \mathbf{s}_h) = 0 \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{V}}_h^k, \tag{18.5a}$$

$$(d_k\mathbf{u}_h, \mathbf{t}) = -\lambda_h(\mathbf{s}_h, \mathbf{t}) \quad \forall \mathbf{t} \in d_k(\boldsymbol{\mathcal{V}}_h^k). \tag{18.5b}$$

This is a problem of the second kind according to the classification of Part 3; thus it can be analysed using the tools of Section 14.

The first step consists in the introduction of suitable operators $T^{(2)}$: $\mathbf{L}^2(\Omega, \Lambda^{k+1}) \to \mathbf{L}^2(\Omega, \Lambda^{k+1})$ and $T_h^{(2)} : \mathbf{L}^2(\Omega, \Lambda^{k+1}) \to d_k(\boldsymbol{\mathcal{V}}_h^k)$, by using the second components of the solutions to the source problems corresponding to (17.6) and (18.5), respectively. For the reader's convenience, these source problems are as follows: given $\mathbf{g} \in \mathbf{L}^2(\Omega, \Lambda^{k+1})$, find $\mathbf{u} \in \mathbf{H}_0(d_k; \Omega)$ and $\mathbf{s} \in \mathbf{K}_{k+1}^d$ such that

$$
\begin{aligned}
(\mathbf{u}, \mathbf{v}) + (d_k \mathbf{v}, \mathbf{s}) = 0 \quad & \forall \mathbf{v} \in \mathbf{H}_0(d_k; \Omega), \\
(d_k \mathbf{u}, \mathbf{t}) = -(\mathbf{g}, \mathbf{t}) \quad & \forall \mathbf{t} \in \mathbf{K}_{k+1}^d,
\end{aligned}
$$

and find $\mathbf{u}_h \in \boldsymbol{\mathcal{V}}_h^k$ and $\mathbf{s}_h \in d_k(\boldsymbol{\mathcal{V}}_h^k)$ such that

$$
\begin{aligned}
(\mathbf{u}_h, \mathbf{v}) + (d_k \mathbf{v}, \mathbf{s}_h) = 0 \quad & \forall \mathbf{v} \in \boldsymbol{\mathcal{V}}_h^k, \\
(d_k \mathbf{u}_h, \mathbf{t}) = -(\mathbf{g}, \mathbf{t}) \quad & \forall \mathbf{t} \in d_k(\boldsymbol{\mathcal{V}}_h^k),
\end{aligned}
$$

respectively.

The theory of Section 14 uses the following spaces: $\mathcal{U}_0 \subset \mathbf{H}_0(d_k; \Omega)$ and $\mathcal{S}_0 \subset \mathbf{K}_{k+1}^d$ denote the spaces containing all solutions $\mathbf{u}$ and $\mathbf{s}$, respectively, of the continuous source problem when the datum $\mathbf{g}$ varies in $\mathbf{L}^2(\Omega, \Lambda^{k+1})$; the discrete kernel of the operator d_k is given by

$$
\mathbb{K}_h = \{\mathbf{v} \in \boldsymbol{\mathcal{V}}_h^k : (d_k \mathbf{v}, \mathbf{t}) = 0 \ \forall \mathbf{t} \in d_k(\boldsymbol{\mathcal{V}}_h^k)\},
$$

and in this particular case it is clearly included in the continuous kernel, that is, $d_k \mathbf{v} = 0$ for all $\mathbf{v} \in \mathbb{K}_h$. In this setting, the three fundamental conditions for the convergence of the eigensolution of (18.5) towards those of (17.6) are the *weak approximability* of $\mathcal{S}_0$, the *strong approximability* of $\mathcal{S}_0$, and the *Fortid condition* (see Definitions 14.1, 14.2 and 14.5, respectively).

The weak approximability requires the existence of $\rho_W(h)$, tending to zero as h tends to zero, such that

$$
(d_k \mathbf{v}, \mathbf{t}) \le \rho_W(h) \|\mathbf{v}\|_{\mathbf{L}^2(\Omega, \Lambda^k)} \|\mathbf{t}\|_{\mathcal{S}_0}. \tag{18.6}
$$

The strong approximability means that there exists $\rho_S(h)$, tending to zero as h tends to zero, such that, for any $\mathbf{s} \in \mathcal{S}_0$, there exists $\mathbf{s}^I \in d_k(\boldsymbol{\mathcal{V}}_h^k)$ with

$$
\|\mathbf{s} - \mathbf{s}^I\|_{\mathbf{L}^2(\Omega, \Lambda^{k+1})} \le \rho_S(h) \|\mathbf{s}\|_{\mathcal{S}_0}. \tag{18.7}
$$

The last property is related to the Fortin operator, that is, an operator $\Pi_h : \mathcal{U}_0 \to \boldsymbol{\mathcal{V}}_h^k$ such that

$$
\begin{aligned}
(d_k(\mathbf{u} - \Pi_h \mathbf{u}), \mathbf{t}) = 0 \quad & \forall \mathbf{u} \in \mathcal{U}_0 \ \forall \mathbf{t} \in d_k(\boldsymbol{\mathcal{V}}_h^k), \\
\|\Pi_h \mathbf{u}\|_{\mathbf{L}^2(\Omega, \Lambda^k)} \le C \|u\|_{\mathcal{U}_0} \quad & \forall \mathbf{u} \in \mathcal{U}_0.
\end{aligned}
$$

The Fortid property expresses the existence of $\rho_F(h)$, tending to zero as h tends to zero, such that

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_{\mathbf{L}^2(\Omega,\Lambda^k)} \leq \rho_F(h)\|\mathbf{u}\|_{\mathcal{U}_0}. \tag{18.8}$$

The next proposition is the analogue of Theorem 14.6 in the setting of this section.

Proposition 18.2. If the weak approximability of $\mathcal{S}_0$ (18.6), the strong approximability of $\mathcal{S}_0$ (18.7), and the Fortid property (18.8) hold true, then there exists $\rho(h)$, tending to zero as h tends to zero, such that

$$\|(T^{(2)} - T_h^{(2)})\mathbf{g}\|_{\mathbf{L}^2(\Omega,\Lambda^{k+1})} \leq \rho(h)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega,\Lambda^{k+1})}.$$

19. Discrete compactness property

We introduce the approximation of problem (17.1) as follows: find $\lambda_h \in \mathbb{R}$ and $\mathbf{u}_h \in \boldsymbol{\mathcal{V}}_h^k \subset \mathbf{H}_0(d_k;\Omega)$ with $\mathbf{u} \neq 0$ such that

$$(d_k\mathbf{u}_h, d_k\mathbf{v}) = \lambda_h(\mathbf{u}_h, \mathbf{v}) \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{V}}_h^k. \tag{19.1}$$

The results of Section 18 can be used for the analysis of (19.1) with the following strategy. Proposition 17.4 states that all solutions to problem (17.1) with positive frequency are in one-to-one correspondence with the solutions of problems (17.4) and (17.6). If a similar result is true for the discrete solutions of (19.1), (18.1) and (18.5), then we can use the theory of the approximation of mixed formulations.

The equivalence of the discrete problems is true in the setting of the de Rham diagram (16.1). Indeed, the much weaker assumption

$$d_{k-1}(\boldsymbol{\mathcal{V}}_h^{k-1}) \subset \boldsymbol{\mathcal{V}}_h^k$$

is sufficient for proving the following result, which is the discrete version of Proposition 17.4.

Proposition 19.1. Let $(\lambda_h, \mathbf{u}_h) \in \mathbb{R} \times \boldsymbol{\mathcal{V}}_h^k$ be a solution of (19.1) with $\lambda_h > 0$; then $(\lambda_h, \mathbf{u}_h)$ is the solution of (18.1) and there exists $\mathbf{s}_h \in d_k(\boldsymbol{\mathcal{V}}_h^k)$ such that $(\lambda_h, \mathbf{s}_h)$ is the solution of (18.5). Conversely, if $(\lambda_h, \mathbf{u}_h) \in \mathbb{R}\times\boldsymbol{\mathcal{V}}_h^k$ is a solution of (18.1), then $\lambda_h > 0$ and $(\lambda_h, \mathbf{u}_h)$ solves (19.1); if $(\lambda_h, \mathbf{s}_h) \in \mathbb{R} \times d_k(\boldsymbol{\mathcal{V}}_h^k)$ is a solution of (18.5) for some $\mathbf{u}_h \in \boldsymbol{\mathcal{V}}_h^k$, then $(\lambda_h, \mathbf{u}_h)$ solves (19.1).

In this section we show how it is possible to introduce suitable conditions that ensure the convergence of the solutions of (19.1) towards those of (17.1). The main condition is the so-called discrete compactness property. Given a finite-dimensional subspace $\boldsymbol{\mathcal{V}}_h^k$ of $\mathbf{H}_0(d_k;\Omega)$, we introduce the subspace of *discretely co-closed* k-forms,

$$\boldsymbol{\mathcal{Z}}_h^k = \{\mathbf{v} \in \boldsymbol{\mathcal{V}}_h^k : (\mathbf{v}, \mathbf{w}) = 0 \ \forall \mathbf{w} \in \boldsymbol{\mathcal{V}}_h^k \text{ with } d_k\mathbf{w} = 0\}.$$

In the case of the de Rham diagram (16.1) and trivial cohomologies, the space $\mathcal{Z}_h^k$ can also be expressed in terms of orthogonalities with respect to $d_{k-1}(\mathcal{V}_h^{k-1})$; more precisely,

$$\mathcal{Z}_h^k = \{\mathbf{v} \in \mathcal{V}_h^k : (\mathbf{v}, d_{k-1}\mathbf{q}) = 0 \ \forall \mathbf{q} \in \mathcal{V}_h^{k-1}\}.$$

If the cohomologies are not trivial, then the two descriptions of $\mathcal{Z}_h^k$ differ by a finite-dimensional space; in particular, the following definition can easily be proved invariant from this choice.

Definition 19.2. We say that the *discrete compactness property* holds for a family $\{\mathcal{V}_h^k\}_h$ of finite-dimensional subspaces of $\mathbf{H}_0(d_k;\Omega)$ if any sequence $\{\mathbf{u}_n\} \subset \mathbf{H}_0(d_k;\Omega)$, with $\mathbf{u}_n \in \mathcal{Z}_{h_n}^k$, which is bounded in $\mathbf{H}_0(d_k;\Omega)$ contains a subsequence which converges in $\mathbf{L}^2(\Omega, \Lambda^k)$.

Remark 19.3. The definition of discrete compactness is often found in the literature with the following formulation: ... *the discrete compactness property holds ... if any sequence* $\{\mathbf{u}_h\}$*, with* $\mathbf{u}_h \in \mathcal{Z}_h^k$*, which is bounded in* $\mathbf{H}_0(d_k;\Omega)$ *contains a subsequence which converges in* $\mathbf{L}^2(\Omega, \Lambda^k)$. Here we make it explicit that the sequence $\mathbf{u}_n$ refers to an arbitrary index choice h_n. This is needed to avoid abstract situations occurring in cases such as when the family $\{\mathcal{V}_h^k\}_h$ comprises *good* spaces interspersed with an infinite number of *bad* spaces. Without extracting the first arbitrary subsequence associated with h_n, the negative effect of the *bad* spaces might be annihilated by a suitable subsequence choice (Christiansen 2009).

It can easily be shown that the limit $\mathbf{u}$ of the subsequence appearing in Definition 19.2 is in $\mathbf{H}_0(d_k;\Omega)$, and that $\delta_k\mathbf{u} = 0$ whenever $d_{k-1}(\mathcal{V}_h^{k-1})$ provides a good approximation of $d_{k-1}(\mathbf{H}_0(d_{k-1};\Omega))$. This motivates the following definition.

Definition 19.4. We say that the *strong compactness property* holds for a family $\{\mathcal{V}_h^k\}_h$ of finite-dimensional subspaces of $\mathbf{H}_0(d_k;\Omega)$ if it satisfies the discrete compactness property, and the limit $\mathbf{u}$ of the subsequence appearing in Definition 19.2 is a co-closed form, that is, $\delta_k\mathbf{u} = 0$.

Remark 19.5. It is worth noticing that, in general, the strong discrete compactness property is not 'much stronger' than the standard discrete compactness property. Indeed, if the space sequence $\{\mathcal{V}_h^{k-1}\}$ has good approximation properties and the discrete compactness property holds for $\{\mathcal{V}_h^k\}$, then passing to the limit in the orthogonality $(\mathbf{v}, d_{k-1}\mathbf{q}) = 0$ which defines the space $\mathcal{Z}_h^k$ gives the strong discrete compactness.

The main result of this section is stated in the next theorem. We consider k-forms in $\mathbf{H}_0(d_k;\Omega)$ and a sequence of finite element spaces $\mathcal{V}_h^k \subset \mathbf{H}_0(d_k;\Omega)$. Moreover, we suppose that we can write problems (17.4) and (17.6), that is, we have $\mathbf{H}_0(d_{k-1};\Omega)$, $\mathbf{K}_{k+1}^d$, and their approximations $\mathcal{V}_h^{k-1}$ and $d_k(\mathcal{V}_h^k)$.

Theorem 19.6. The following three sets of conditions are equivalent.

(i) The strong discrete compactness property (see Definition 19.4) and the following standard approximation property: for any $\mathbf{v} \in \mathbf{H}_0(d_k;\Omega)$ with $\delta_k\mathbf{v} = 0$, there exists a discrete sequence $\{\mathbf{v}_h\} \subset \boldsymbol{\mathcal{V}}_h^k$ such that

$$\|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{H}(d_k;\Omega)} \to 0 \quad \text{as } h \to 0.$$

(ii) The ellipticity in the kernel (18.2), the weak approximability of $\mathcal{Q}_0$ (18.3), and the strong approximability of $\mathcal{V}_0$ (18.4).

(iii) The weak approximability of $\mathcal{S}_0$ (18.6), the strong approximability of $\mathcal{S}_0$ (18.7), and the existence of a Fortin operator satisfying the Fortid property (18.8).

Proof. The proof is a generalization of the result of Boffi (2007, Theorem 3) where it is split into a series of propositions. We report here the main arguments for the sake of completeness.

Let us start with the implication (i) $\Rightarrow$ (ii).

The strong discrete compactness property implies the ellipticity in the kernel by the generalization of Monk and Demkowicz (2001, Corollary 4.2).

The strong discrete compactness property implies the weak approximability of $\mathcal{Q}_0$ from the following argument. By contradiction, let $\{\mathbf{q}_h\} \subset \mathcal{Q}_0$ be a sequence such that there exists $\{\mathbf{v}_h\} \subset \mathbb{K}_h$ with

$$\|\mathbf{q}_h\|_{\mathcal{Q}_0} = 1,$$
$$\|\mathbf{v}_h\|_{\mathbf{H}(d_k;\Omega)} = 1,$$
$$(\mathbf{v}_h, d_{k-1}\mathbf{q}_h) \geq \varepsilon_0 > 0.$$

From the boundedness of $\{\mathbf{q}_h\}$ and the strong discrete compactness, we can extract subsequences (denoted with the same notation) $\{\mathbf{q}_h\}$ and $\{\mathbf{v}_h\}$, and there exist $\mathbf{q} \in \mathbf{H}_0(d_{k-1};\Omega)$ and $\mathbf{v} \in \mathbf{L}^2(\Omega,\Lambda^k)$ such that $\mathbf{q}_h \rightharpoonup \mathbf{q}$ weakly in $\mathbf{H}_0(d_{k-1};\Omega)$ and $\mathbf{v}_h \to \mathbf{v}$ strongly in $\mathbf{L}^2(\Omega,\Lambda^k)$. Moreover, $\delta_k\mathbf{v} = 0$. Passing to the limit gives

$$(\mathbf{v}, d_{k-1}\mathbf{q}) \geq \varepsilon_0,$$

which contradicts $\delta_k\mathbf{v} = 0$.

The strong approximability of $\mathcal{V}_0$ is a consequence of (i) by the following argument. By contradiction we assume that the strong approximability of $\mathcal{V}_0$ is not satisfied. Let $\{\mathbf{u}_n\} \subset \mathcal{V}_0$ be a sequence such that

$$\|\mathbf{u}_n\|_{\mathcal{V}_0} = 1,$$
$$\inf_{\mathbf{v}_{h_n} \in \mathbb{K}_{h_n}} \|\mathbf{u}_n - \mathbf{v}_{h_n}\|_{\mathbf{H}(d_k;\Omega)} \geq \varepsilon_0 > 0 \quad \forall n,$$

where h_n is a sequence of mesh sizes tending to zero. From the compact embedding of $\mathcal{V}_0$ in $\mathbf{H}_0(d_k;\Omega)$, it follows that up to a subsequence there exists $\mathbf{u} \in \mathbf{H}_0(d_k;\Omega)$ such that $\mathbf{u}_n \to \mathbf{u}$ in $\mathbf{H}(d_k;\Omega)$. Moreover, we have $\delta_k \mathbf{u} = 0$. We reach a contradiction if we are able to approximate $\mathbf{u}$ in $\mathbf{H}(d_k;\Omega)$ with a sequence in $\mathbb{K}_{h_n}$. From the approximation property in (i) there exists $\mathbf{u}_{h_n} \subset \boldsymbol{\mathcal{V}}^k_{h_n}$ such that $\mathbf{u}_{h_n} \to \mathbf{u}$ in $\mathbf{H}(d_k;\Omega)$. We perform a discrete Hodge decomposition,

$$\mathbf{u}_{h_n} = d_{k-1}\mathbf{p}_{h_n} + \mathbf{u}^I_{h_n},$$

as follows. We take $\mathbf{p}_{h_n} \in \boldsymbol{\mathcal{V}}^{k-1}_{h_n}$ such that

$$(d_{k-1}\mathbf{p}_{h_n}, d_{k-1}\mathbf{q}) = (\mathbf{u}_{h_n}, d_{k-1}\mathbf{q}) \quad \forall \mathbf{q} \in \boldsymbol{\mathcal{V}}^{k-1}_{h_n}$$

(this $\mathbf{p}_{h_n}$ is not unique if $k > 1$), and define $\mathbf{u}^I_{h_n}$ by difference. By definition $\{\mathbf{u}^I_{h_n}\}$ is bounded in $\mathbf{H}(d_k;\Omega)$ and belongs to $\boldsymbol{\mathcal{Z}}^k_{h_n}$, so that up to a subsequence it converges in $\mathbf{L}^2(\Omega,\Lambda^k)$ to a limit $\mathbf{u}^*$ satisfying $\delta_k\mathbf{u}^* = 0$. Since $\{\mathbf{u}^I_{h_n}\}$ belongs to $\mathbb{K}_{h_n}$, it is enough to prove that $\mathbf{u} = \mathbf{u}^*$. We have that $\mathbf{u} - \mathbf{u}^* = d_{k-1}\mathbf{p}$, with $\mathbf{p} \in \mathbf{H}_0(d_{k-1};\Omega)$ and $d_{k-1}\mathbf{p}_{h_n} \to d_{k-1}\mathbf{p}$ in $\mathbf{L}^2(\Omega,\Lambda^k)$. On the other hand, $\delta_k(\mathbf{u}-\mathbf{u}^*) = 0$ and $\mathbf{u}-\mathbf{u}^* = d_{k-1}\mathbf{p}$ imply $\mathbf{u}-\mathbf{u}^* = 0$.

We now consider the implication (ii) ⇒ (iii).

The hypotheses in (ii), according to Proposition 18.1, imply that the eigensolutions of (18.1) converge towards those of (17.4). From the equivalences stated in Propositions 17.4 and 19.1, it follows that the eigensolutions of (18.5) converge towards those of (17.6) as well. Hence, the conditions in (iii) are satisfied since they are necessary for the norm convergence of $T^{(2)}_h$ to $T^{(2)}$ in $\mathcal{L}(\mathbf{L}^2(\Omega,\Lambda^{k+1}), \mathbf{K}^d_{k+1})$ (Boffi *et al.* 1997, Theorem 7).

Finally, let us show that (iii) ⇒ (i).

First we prove that (iii) implies the strong discrete compactness property. Let $\{\mathbf{u}_n\}$ be a sequence as in Definition 19.2. It follows that $\mathbf{u}_n$ satisfies

$$(\mathbf{u}_n, \mathbf{v}) + (d_k\mathbf{v}, \mathbf{s}_n) = 0 \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{V}}^k_{h_n} \tag{19.2}$$

for a suitable $\mathbf{s}_n \in d_k(\boldsymbol{\mathcal{V}}^k_{h_n})$. We define $\mathbf{u}(n) \in \mathbf{H}_0(d_k;\Omega)$ and $\mathbf{s}(n) \in \mathbf{K}^d_{k+1}$ by

$$(\mathbf{u}(n), \mathbf{v}) + (d_k\mathbf{v}, \mathbf{s}(n)) = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0(d_k;\Omega), \tag{19.3a}$$

$$(d_k\mathbf{u}(n), \mathbf{t}) = (d_k\mathbf{u}_n, \mathbf{t}) \quad \forall \mathbf{t} \in \mathbf{K}^d_{k+1}. \tag{19.3b}$$

In particular, $\{\mathbf{u}(n)\}$ is bounded in $\mathbf{H}_0(d_k;\Omega) \cap \mathbf{H}(\delta_k;\Omega)$, which is compact in $\mathbf{L}^2(\Omega,\Lambda^k)$, so that there exists a limit $\mathbf{u}$ with $\delta_k\mathbf{u} = 0$ such that $\mathbf{u}(n) \to \mathbf{u}$ in $\mathbf{L}^2(\Omega,\Lambda^k)$ (up to a subsequence). We can show the strong discrete compactness property if we can prove that $\mathbf{u}_n$ tends to $\mathbf{u}$ in $\mathbf{L}^2(\Omega,\Lambda^k)$. The conditions in (iii) guarantee the norm convergence of $T^{(2)}_h$ to $T^{(2)}$ in $\mathcal{L}(\mathbf{L}^2(\Omega,\Lambda^{k+1}), \mathbf{L}^2(\Omega,\Lambda^{k+1}))$, which implies

$$\|\mathbf{s}(n) - \mathbf{s}_n\|_{\mathbf{L}^2(\Omega,\Lambda^{k+1})} \le \rho(n)\|d_k\mathbf{u}_n\|_{\mathbf{L}^2(\Omega,\Lambda^{k+1})},$$

with $\rho(n)$ tending to zero as n tends to infinity. Choosing $\mathbf{v} = \mathbf{u}(n) - \mathbf{u}_n$

in (19.3) gives

$$(\mathbf{u}(n), \mathbf{u}(n) - \mathbf{u}_n) = 0,$$

and the difference between the first equation in (19.3) and (19.2) leads to

$$-(\mathbf{u}(n) - \mathbf{u}_n, \mathbf{v}) + (d_k\mathbf{v}, \mathbf{s}(n) - \mathbf{s}_n) = 0 \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{V}}^k_{h_n}.$$

The last two equations and the choice $\mathbf{v} = \mathbf{u}_n$ give

$$\begin{aligned}\|\mathbf{u}(n) - \mathbf{u}_n\|^2_{\mathbf{L}^2(\Omega,\Lambda^k)} &\le \|\mathbf{s}(n) - \mathbf{s}_n\|_{\mathbf{L}^2(\Omega,\Lambda^{k+1})}\|d_k\mathbf{u}_n\|_{\mathbf{L}^2(\Omega,\Lambda^{k+1})} \\ &\le \rho(n)\|d_k\mathbf{u}_n\|^2_{\mathbf{L}^2(\Omega,\Lambda^{k+1})},\end{aligned}$$

which implies $\mathbf{u}_n \to \mathbf{u}$ in $\mathbf{L}^2(\Omega, \Lambda^k)$. Finally, the approximation property in (i) follows from the fact that (iii) implies the correct approximation of the eigenfunctions of (17.6), which is equivalent to (17.4) in the spirit of Proposition 17.4. Since the eigenfunctions are a dense space in the set of functions $\mathbf{v}$ in $\mathbf{H}(d_k; \Omega)$ with $\delta_k\mathbf{v} = 0$, the approximation property follows from the approximation of the eigenfunctions. □

The main consequence of Theorem 19.6 is that the discrete compactness property and standard approximabilities are designated as the natural conditions for good convergence of the eigensolutions of (19.1) towards those of (17.1). In the next section we are going to show how this theory can be applied to the approximation of Maxwell's eigenvalue problem.

20. Edge elements for the approximation of Maxwell's eigenvalue problem

We conclude this part with a discussion about the relationships between the results presented so far and the approximation of Maxwell's eigenvalue problem, which has been the main motivation for the author's study of the finite element approximation of eigenvalue problems in the setting of differential forms.

In Section 5 we recalled the definition of Maxwell's eigenvalue problem and presented some numerical examples concerning its approximation. We explained how edge finite elements are the correct choice for the discretization of problem (5.3), which is a particular case of problem (19.1) ($k = 1$). It was also discussed that the direct use of standard (nodal) finite element produces unacceptable results (see, in particular, Figures 5.2, 5.3, 5.4, and Table 5.3 with Figures 5.6 and 5.7). Some modifications of problem (5.3) are available that allow the use of standard finite elements (Costabel and Dauge 2002) or of standard finite elements enriched with bubble functions (Bramble, Kolev and Pasciak 2005).

In this section we review some basic literature about edge finite elements and show how Theorem 19.6 applies to this situation. We also discuss the difference between the discrete compactness property (see Definition 19.2) and the strong discrete compactness property (see Definition 19.4).

Edge finite elements were introduced by Nédélec (1980, 1986). The entire family of mixed finite elements is often referred to as the Nédélec–Raviart–Thomas family, since Raviart–Thomas elements (Raviart and Thomas 1977) also belong to this family. Other families are available: among them we recall Brezzi–Douglas–Marini elements (Brezzi *et al.* 1985, 1986), Brezzi–Douglas–Fortin–Marini elements (Brezzi *et al.* 1987*a*, 1987*b*), and the *hp* adaptive family presented by Demkowicz and co-workers (Vardapetyan and Demkowicz 1999, Demkowicz, Monk, Vardapetyan and Rachowicz 2000*b*). The merit of linking edge elements to the de Rham complex comes from the fundamental work of Bossavit (1988, 1989, 1990, 1998). The idea is intrinsically related to the concept of Whitney forms (Whitney 1957) and it should be acknowledged that lowest-order edge finite elements are often referred to as Whitney elements. The de Rham complex is a more complete viewpoint than the so-called *commuting diagram property* (Douglas and Roberts 1982), which was introduced in the framework of mixed approximations. Many authors have discussed the relationship between finite elements for electromagnetic problems and differential forms. Among others, the following papers and the references therein give an idea of the underlying discussion: Hiptmair (1999*a*, 2002), Gross and Kotiuga (2004), Christiansen (2007) and Boffi (2001). A deep understanding of the subject which leads to the formalism of the finite element exterior calculus is presented in the following works: Arnold (2002) and Arnold *et al.* (2006*a*, 2006*b*, 2010).

Proposition 17.4 in the context of Maxwell's eigenvalue problem states that there are three equivalent formulations.

(1) The standard Maxwell eigenvalue problem: find $\lambda \in \mathbb{R}$ and $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ with $\mathbf{u} \neq 0$ such that

$$(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v}) = \lambda(\boldsymbol{\varepsilon}\mathbf{u},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl};\Omega).$$

(2) The mixed formulation of the first type (Kikuchi 1987): find $\lambda \in \mathbb{R}$ and $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ with $\mathbf{u} \neq 0$ such that, for $p \in H^1_0(\Omega)$,

$$\begin{aligned}(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v}) + (\boldsymbol{\varepsilon}\mathbf{v},\mathbf{grad}\,p) &= \lambda(\boldsymbol{\varepsilon}\mathbf{u},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl};\Omega),\\ (\boldsymbol{\varepsilon}\mathbf{u},\mathbf{grad}\,q) &= 0 \quad \forall q \in H^1_0(\Omega).\end{aligned}$$

(3) The mixed formulation of the second type (Boffi *et al.* 1999*b*): find $\lambda \in \mathbb{R}$ and $\mathbf{s} \in \Sigma$ with $\mathbf{s} \neq 0$ such that, for $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)$,

$$\begin{aligned}(\boldsymbol{\varepsilon}\mathbf{u},\mathbf{v}) + (\boldsymbol{\mu}^{-1/2}\,\mathbf{curl}\,\mathbf{v},\mathbf{s}) &= 0 \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl};\Omega),\\ (\boldsymbol{\mu}^{-1/2}\,\mathbf{curl}\,\mathbf{u},\mathbf{t}) &= -\lambda(\mathbf{s},\mathbf{t}) \quad \forall \mathbf{t} \in \Sigma,\end{aligned}$$

with $\Sigma = \boldsymbol{\mu}^{-1/2}\,\mathbf{curl}(\mathbf{H}_0(\mathbf{curl};\Omega))$.

From Proposition 19.1 the equivalence holds at the discrete level as well. We would like to point out that, from the computational point of view, the standard formulation is the most convenient and is the one commonly used. The two mixed formulations have essentially been introduced for the theoretical analysis of the finite element approximation. Some comments on the computational issues can be found, for instance, in Simoncini (2003) and Arbenz and Geus (1999). For multigrid solvers, the reader is referred to Hiptmair (1999*b*), Arnold, Falk and Winther (2000), Reitzinger and Schöberl (2002), and to the references therein.

Theorem 19.6 is the main tool for the analysis of the problem we are interested in. In particular, conditions stated in items (i), (ii) and (iii) refer to the standard formulation, the first mixed formulation, and the second mixed formulation, respectively. In the literature, conditions (i) and (iii) have mostly been used. Condition (i) was used in Boffi, Conforti and Gastaldi (2006*a*) for the analysis of a modification of Maxwell's eigenvalue problem for the approximation of band gaps in photonic crystals.

The case of a two-dimensional domain $\Omega \subset \mathbb{R}^2$ can easily be analysed. For instance, using the second mixed formulation and the equivalence of rot and div operators, we are led to the problem of approximating the eigensolutions of the Neumann problem for the Laplace operator in mixed form with Raviart–Thomas elements (see Section 5.1). The analysis of this problem was performed by Falk and Osborn (1980) (see also Demkowicz, Monk, Schwab and Vardapetyan (2000*a*)).

For a three-dimensional domain $\Omega \subset \mathbb{R}^3$, the discrete compactness property was proved by Kikuchi (1989) in the case of lowest-order tetrahedral elements. The convergence of the h method for practically all known families of edge finite elements follows from the arguments of Boffi (2000), where the Fortid condition, which makes it possible to use the second mixed formulation (see Theorem 19.6(iii)), was proved. A direct proof of convergence of the eigensolutions, which makes use of the discrete compactness property and of the abstract results of Anselone (1971), was given by Monk and Demkowicz (2001) under the assumption of quasi-uniformity of the mesh (see also Ciarlet and Zou (1999), Hiptmair (2002), Monk (2003) and Costabel and Dauge (2003)).

In the work of Caorsi, Fernandes and Raffetto (2000) (see also Caorsi, Fernandes and Raffetto (2001)) it was proved that the discrete compactness property is a necessary and sufficient condition for good convergence of the eigensolutions of Maxwell's system. In that paper the strong discrete compactness is not explicitly addressed, but suitable approximation properties are considered (see Remark 19.5)

All the results presented so far fit very well into the theoretical setting of Arnold *et al.* (2010), where a more general theory is developed for the analysis of the approximation of the Hodge–Laplace eigenvalue problem (17.2).

In that very recent paper it has been proved that a sufficient condition for the convergence of the eigensolution is the existence of projection operators that are uniformly bounded in $\mathbf{L}^2(\Omega, \Lambda^k)$. A construction of such operators is presented (for any admissible k and n), which is performed by means of a suitable extension–regularization procedure. It is also shown that the existence of such projections implies the Fortid property and the discrete compactness property.

In practical applications the computational domains and the material properties related to Maxwell's cavities are such that the eigensolutions often correspond to non-smooth eigenfunctions. It has been observed that in such cases it may be convenient to use anisotropic elements (Nicaise 2001, Buffa, Costabel and Dauge 2005), suitable spaces that take care of the singular functions (Assous, Ciarlet and Sonnendrücker 1998), or an adaptive *hp* strategy (Demkowicz 2005, Ainsworth and Coyle 2003, Hiptmair and Ledger 2005).

It would be nice to construct suitable projections which are bounded in $\mathbf{L}^2(\Omega, \Lambda^k)$, uniformly in p, so that the theory of Arnold *et al.* (2010) can be applied to the analysis of the p and perhaps of the *hp* version of edge finite elements. Unfortunately, such projections are not yet available, and it is not clear whether they exist.

The first result concerning the *hp* version of edge finite elements was by Boffi, Demkowicz and Costabel (2003), where the triangular case was analysed and the discrete compactness property was proved modulo a conjectured estimate, which was only demonstrated numerically. The first rigorous proof of the discrete compactness property for the *hp* version of edge elements was by Boffi, Costabel, Dauge and Demkowicz (2006*b*) in the case of rectangular meshes allowing for one-irregular hanging nodes. It is interesting to note that the plain p version of edge finite elements (pure spectral elements) had never been analysed before that paper, even though there was evidence of good performance (Wang, Monk and Szabo 1996). Finally, a fairly general result concerning the p version of edge finite elements can be found in Boffi *et al.* (2009), where the discussion is performed in the framework of differential forms, and where the role of the discrete compactness property has also been studied.

We conclude this presentation with a comment on the role of the discrete compactness property and the strong discrete compactness property. In particular, we want to emphasize the differences between the two conditions (see Boffi (2007, Section 5) and Boffi *et al.* (2009, Section 2.3)). The behaviour we are going to describe is strictly related to the concepts of spectral correctness and spurious-free approximation introduced in Caorsi *et al.* (2000, Section 4). We go back to the notation of the h version of finite elements, but our comments apply to the p and *hp* versions as well. It is clear that the main difficulties in the approximation of problem (17.1) come from the discretization of the infinite-dimensional kernel that occurs for $k \geq 1$. The discrete compactness property is related to the spectrally correct approximation, that is, all continuous eigenvalues (including the zero frequency) are approximated by a correct number of discrete eigenvalues and the corresponding eigenspaces are well approximated. For $k = 0$, in the case of a compact resolvent, this is an optimal notion, and implies that the numerical scheme is capable of providing a good approximation of the eigenmodes. For $k \geq 1$, however, the eigenvalues approximating the infinite-dimensional kernel may pollute the whole spectrum, and the numerical scheme becomes unusable. Moreover, Caorsi *et al.* (2000) showed that, if $\mathbf{H}_0(d_k;\Omega)$ is well approximated by $\boldsymbol{\mathcal{V}}_h^k$, then the eigenvalues approximating the zero frequency are confined to a region close to zero, which can be made arbitrarily small, for sufficiently small h. The big improvement given by strong discrete compactness (or, analogously, by discrete compactness and completeness of the discrete kernel (CDK) of Caorsi *et al.* (2000)), consists in the conclusion that the discrete frequencies approximating zero are exactly at zero, meaning that all non-physical frequencies are well separated from the rest of the spectrum.

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